3D Lattice Walks Confined to an Octant: Nonrationality of the Second Critical Exponent

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joint work with Luc Hillairet and Kilian Raschel

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CanaDAM

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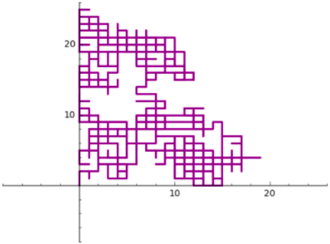
Introduction

Let u_n be the # of walks from 0 to P staying in $C \subset \mathbb{Z}^d$ using n steps from $\mathcal{S} \subset \mathbb{Z}^d$.

$$U(t) = \sum_{n \geq 0} u_n t^n$$

Is $U(t)$ algebraic?
If not, is it D-finite?



Simple random walk with 1000 steps

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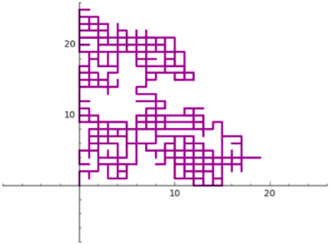


Simple random walk with 1000 steps

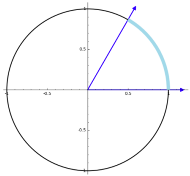
Theorem (1)

If $u_n \sim K \cdot \rho^n \cdot n^\alpha$ and α is irrational, then $U(t)$ is not D-finite

(1) See Thm 3 of Bostan-Raschel-Salvy; this is a consequence of work of André, Chudnovski and Katz

One term asymptotics

Denisov and Wachtel proved for a large class of cones C that $u_n \sim K \cdot \rho^n \cdot n^{\alpha_1}$, where



$$\alpha_1 = -\sqrt{\lambda_1 + \left(\frac{d}{2} - 1\right)^2} - 1.$$

λ_1 is the first *Dirichlet eigenvalue* for the *spherical Laplacian* on $T \subseteq S^{d-1}$

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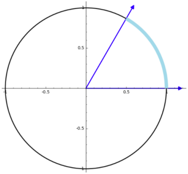

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Definition

λ is an Dirichlet eigenvalue of T if there is $\phi \in C^2(T) \cap C(\overline{T})$ such that

$$\begin{cases} \Delta_{\mathbb{S}^{d-1}} \phi = -\lambda \phi & \text{in } T \\ \phi = 0 & \text{on } \partial T \end{cases}$$

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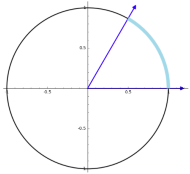

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$$\Delta_{\mathbb{R}^d} f = \sum_{i=1}^{d} \frac{\partial f}{\partial x_i^2} \qquad \Delta_{\mathbb{R}^d} = \frac{\partial^2 f}{\partial r^2} + \frac{d-1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\Delta_{\mathbb{S}^{d-1}} f$$

$$\Delta_{\mathbb{S}^2} f(\theta, \varphi) = \frac{\partial^2 f}{\partial^2 \theta} + \frac{\cos\theta}{\sin\theta}\frac{\partial f}{\partial \theta} + \frac{1}{\sin^2\varphi}\frac{\partial f}{\partial \varphi}$$

One term asymptotics

$$\begin{cases} \Delta_{S^{d-1}}\phi = -\lambda\phi & \text{in } T \\ \phi = 0 & \text{in } \partial T \end{cases}$$

The eigenvalues consist of an infinite sequence

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \to \infty$$

Corresponding to each λ_j are eigenfunctions ϕ_j

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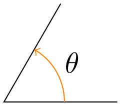
$T := C \cap \mathbb{S}^{d-1}$
$d = 2$: T is an arc



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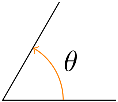
$T := C \cap \mathbb{S}^{d-1}$
$d = 2$: T is an arc
$d = 3$: T is a spherical triangle



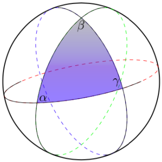


Image credit: BPRT

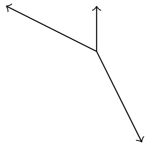
Note: Denisov & Wachtel's result is from probability theory; this consequence of their result appears in Bostan-Raschel-Salvy

Motivation for moving to the continuous setting

D-finite $\Rightarrow$ all α_i's are rational

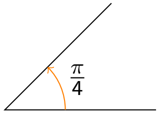
Conjecture. There are models for which α_1 is rational but the generating function is not D-finite.

Ex. (Bostan, Bousquet-Mélou, Melczer)



$\mathcal{S} = \{(-2,1),(0,1),(1,-2)\}$

T is the arc:



$\lambda_1 = 16$; $\alpha_1 = -5$

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Conjecture. There are models for which α_1 is rational but the generating function is not D-finite.

Question. Can we have $\alpha_1 \in \mathbb{Q}$ but $\alpha_2 \notin \mathbb{Q}$?

Denisov and Wachtel $\Rightarrow$ α_1

We move to the continuous setting.

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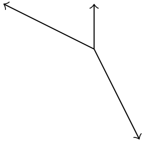

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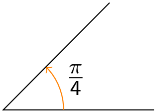

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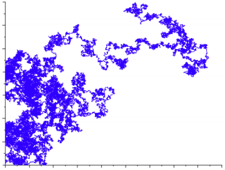
Exit time of Brownian motion from a cone

- Let B_t be a Brownian motion with $B_0 = x$
- Let $\tau = \inf\{t : B_t \notin C\}$
- If $A \subset C$, then

$$\mathbb{P}_x\{B_t \in A, \tau > t\} = \int_A p^C(x, y; t)dy$$



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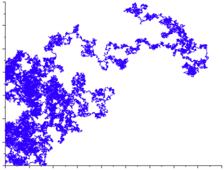
$$\Uparrow$$

continuous analogue of u_n



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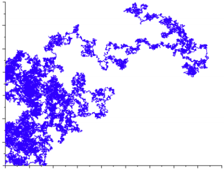

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- Bañuelos and Smits gave an explicit expression for $p^C(x, y; t)$

The heat kernel

Theorem

The heat kernel $p^C(x, y; t)$ admits the complete asymptotic expansion

$$p^C(x, y; t) = K_1 \cdot t^{-\alpha_1} + K_2 \cdot t^{-\alpha_2} + \cdots + K_p \cdot t^{-\alpha_p} + o(t^{-\alpha_p}),$$

where

- K_i *depend on* x *and* y
- α_i *are independent of* x *and* y, $\alpha_1 < \alpha_2 < \cdots < \alpha_p$
- $\alpha_i = \sqrt{\lambda_j + \left(\frac{d}{2} - 1\right)^2} + k, \ k \in \mathbb{N}$
- λ_j*'s are Dirichlet eigenvalues on* $C \cap \mathbb{S}^{d-1}$

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Question. Is it possible to have $\alpha_1, \ldots, \alpha_{p-1}$ rational and α_p irrational?

Main Result

Theorem (Hillairet, J., Raschel)

There exists a 3D cone such that the heat kernel admits the asymptotics with $\alpha_1, \ldots, \alpha_{p-1} \in \mathbb{Q}$ *and* $\alpha_p \notin \mathbb{Q}$.

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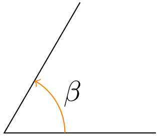
Remark. In 2D, $\alpha_i \in \mathbb{Q}$ or $\alpha_i \notin \mathbb{Q}$ for all i

- $\alpha_i = \sqrt{\lambda_j + \left(\frac{d}{2} - 1\right)^2} + k$
- T is an arc with opening angle β
- The jth eigenvalue of T is $\lambda_j = \left(\frac{\pi j}{\beta}\right)^2$
- Each $\alpha_i = \frac{\pi j}{\beta} + k$, for $j, k \in \mathbb{N}$



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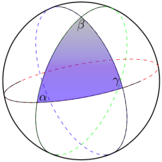
In 3D, T is a spherical triangle

$$\alpha_i = \sqrt{\lambda_j + \frac{1}{4} + k}$$



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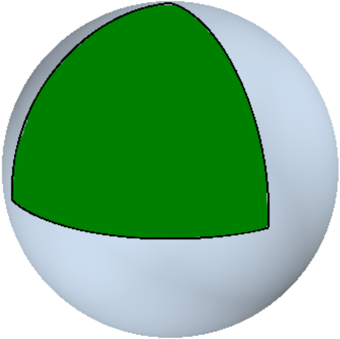
Theorem (Hillairet, J., Raschel)

There exists a one-parameter family of spherical triangles such that

- $\lambda_1(t) = 12 \left(\Rightarrow \alpha_1 = \frac{9}{2} \right)$
- $\lambda_2(t)$ *is real-analytic and non-constant.* $(\Rightarrow \alpha_p$ *nonconstant*$)$

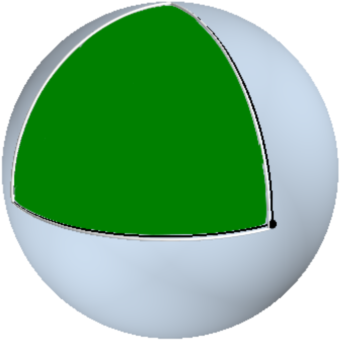
A path of spherical triangles with $\lambda_1 = 12$



Triangle with all angles $\frac{\pi}{2}$

$\lambda_1 = 12, \lambda_2 = 30$

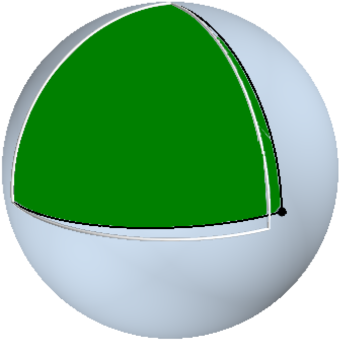
A path of spherical triangles with $\lambda_1 = 12$



$$\lambda_1 = 12.00$$

$$\lambda_2 = 29.768119$$

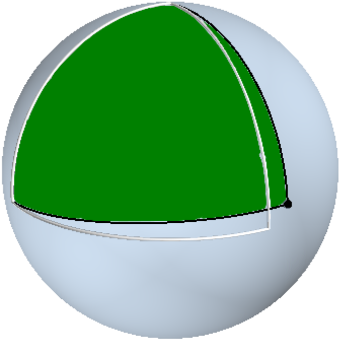
A path of spherical triangles with $\lambda_1 = 12$



$$\lambda_1 = 12.00$$

$$\lambda_2 = 29.040781$$

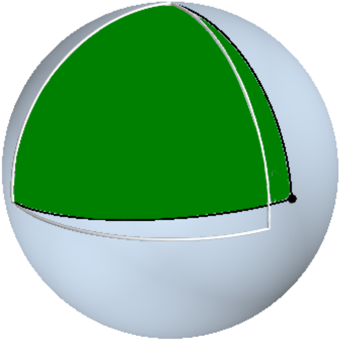
A path of spherical triangles with $\lambda_1 = 12$



$$\lambda_1 = 12.00$$

$$\lambda_2 = 28.618634$$

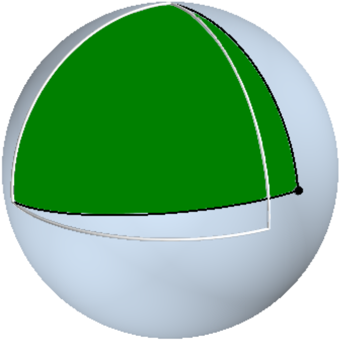
A path of spherical triangles with $\lambda_1 = 12$



$$\lambda_1 = 12.00$$

$$\lambda_2 = 28.314809$$

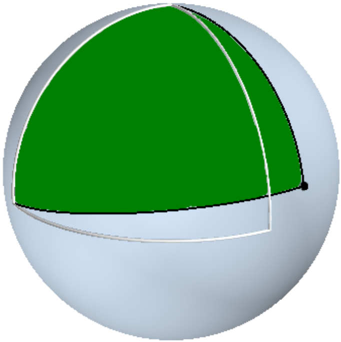
A path of spherical triangles with $\lambda_1 = 12$



$$\lambda_1 = 12.00$$

$$\lambda_2 = 27.848035$$

A path of spherical triangles with $\lambda_1 = 12$



$$\lambda_1 = 12.00$$

$$\lambda_2 = 27.480416$$

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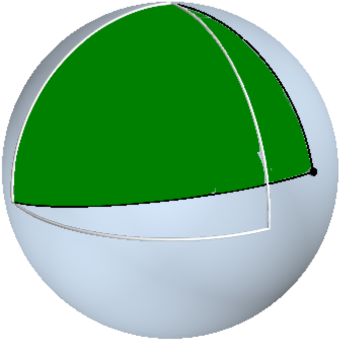
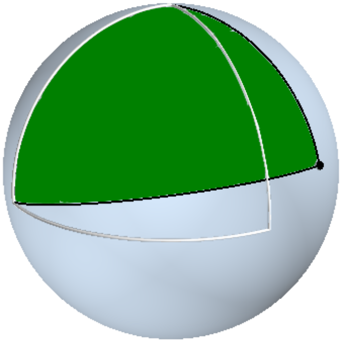


$$\lambda_1 = 12.00$$

$$\lambda_2 = 26.647921$$

A path of spherical triangles with $\lambda_1 = 12$



$$\lambda_1 = 12.00$$

$$\lambda_2 = 26.113383$$

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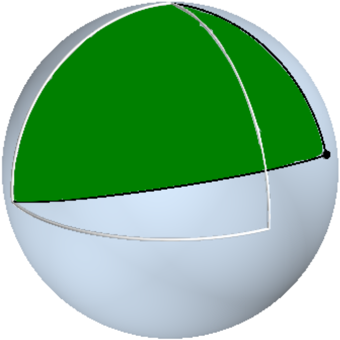
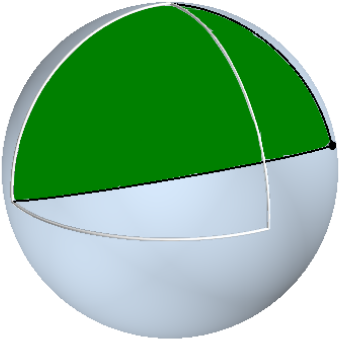


$$\lambda_1 = 12.00$$

$$\lambda_2 = 25.332056$$

A path of spherical triangles with $\lambda_1 = 12$



$$\lambda_1 = 12.00$$

$$\lambda_2 = 24.477623$$

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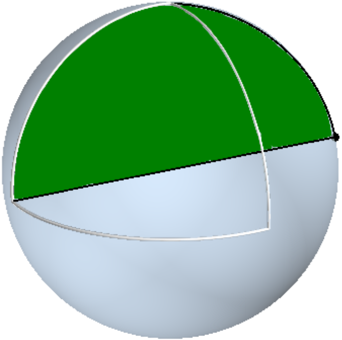
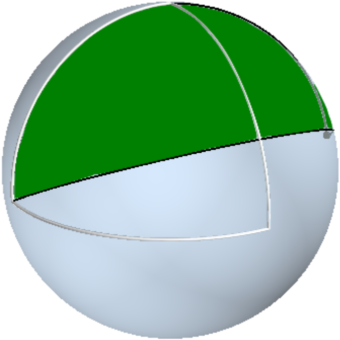


$$\lambda_1 = 12.00$$

$$\lambda_2 = 23.197977$$

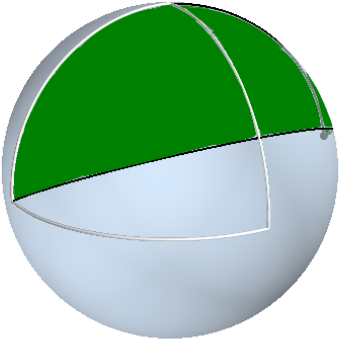
A path of spherical triangles with $\lambda_1 = 12$



$$\lambda_1 = 12.00$$

$$\lambda_2 = 21.703944$$

A path of spherical triangles with $\lambda_1 = 12$



$$\lambda_1 = 12.00$$

$$\lambda_2 = 21.429417$$

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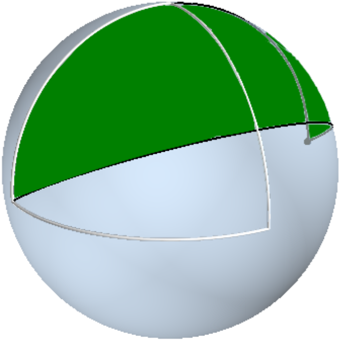
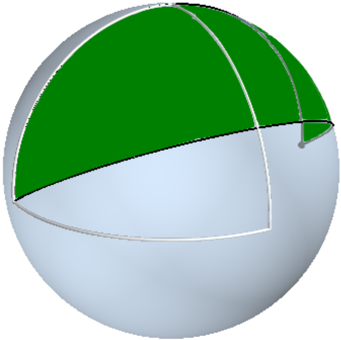


$$\lambda_1 = 12.00$$

$$\lambda_2 = 20.373905$$

A path of spherical triangles with $\lambda_1 = 12$



Digon with angle $\frac{\pi}{3}$

$\lambda_1 = 12, \lambda_2 = 20$

Proof Sketch: Defining the path of spherical triangles

Goal. Find a curve $\gamma(t) = (a(t), b(t))$ on which $\lambda_1(t) = 12$

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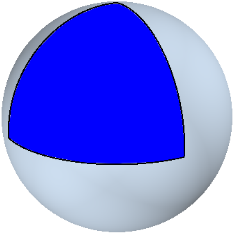
$T_0 :=$ triangle with vertices $(0, 0)$, $(\frac{\pi}{2}, 0)$, $(\frac{\pi}{2}, \frac{\pi}{2})$



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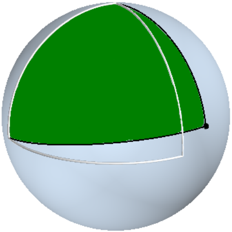

It suffices to consider the tangent vector to the curve of triangles at $t = 0$.

$$\frac{d}{dt}(\gamma(t))|_{t=0} := (a_0, b_0)$$

$T_t :=$ triangle with vertices
$(0,0)$, $(\frac{\pi}{2}, 0)$, $(\frac{\pi}{2} - b_0 t, \frac{\pi}{2} - a_0 t)$

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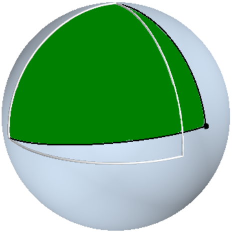

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Dirichlet eigenvalues are functions $\lambda(t)$ such that $\exists\ \phi(t)$ satisfying

$$\begin{cases} \Delta_{\mathbb{S}^2}\phi(t) = -\lambda(t)\phi(t) & \text{in } T_t \\ \phi(t) = 0 & \text{on } \partial T_t \end{cases}$$

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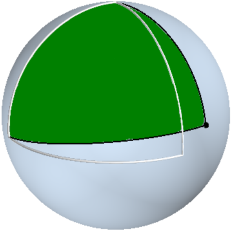
Goal. Find a curve $\gamma(t) = (a(t), b(t))$ on which $\lambda_1(t) = 12$

$T_0 :=$ triangle with vertices $(0,0)$, $(\frac{\pi}{2}, 0)$, $(\frac{\pi}{2}, \frac{\pi}{2})$



It suffices to consider the tangent vector to the curve of triangles at $t = 0$.

$$\frac{d}{dt}(\gamma(t))|_{t=0} := (a_0, b_0)$$

$T_t :=$ triangle with vertices $(0,0)$, $(\frac{\pi}{2}, 0)$, $(\frac{\pi}{2} - b_0 t, \frac{\pi}{2} - a_0 t)$

Dirichlet eigenvalues are functions $\lambda(t)$ such that $\exists\ \phi(t)$ satisfying

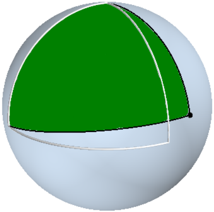
$$\begin{cases} \Delta_{\mathbb{S}^2}\phi(t) = -\lambda(t)\phi(t) & \text{in } T_t \\ \phi(t) = 0 & \text{on } \partial T_t \end{cases}$$

Plan:

- Find a_0, b_0 so that $\frac{d}{dt}(\lambda_1(t))|_{t=0} = 0$
- For this a_0, b_0, $\frac{d}{dt}(\lambda_2(t))|_{t=0} \neq 0$.

Proof Sketch: Variational formulas

Formulas for $\frac{d}{dt}(\lambda_1(t))|_{t=0}$ & $\frac{d}{dt}(\lambda_2(t))|_{t=0}$ come from analytic perturbation theory

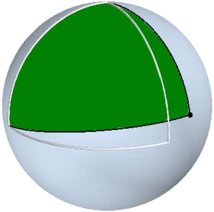


Proof Sketch: Variational formulas

Formulas for $\frac{d}{dt}(\lambda_1(t))|_{t=0}$ & $\frac{d}{dt}(\lambda_2(t))|_{t=0}$ come from analytic perturbation theory

- Strategy: Fix the domain, allow Δ to depend on t.
- Define a diffeomorphism $F_t : T_0 \to T_t$.
- Define g_t to be the pullback metric

$$g_t = F_t^* g_{\mathbb{S}^2}$$



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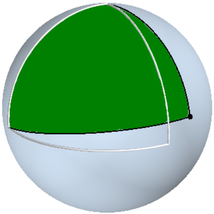

Intuition/example:

- $f : [0,\infty) \times [0, 2\pi) \to \mathbb{R}^2$, $(r,\theta) \mapsto (r\cos\theta, r\sin\theta)$
- $d_{\mathbb{R}^2}((x_1,y_1),(x_2,y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$
- The pullback metric $d_P := f^* d_{\mathbb{R}^2}$ is $d_P((r_1,\theta_1),(r_2,\theta_2)) = \sqrt{(r_1\cos\theta_1 - r_2\cos\theta_2)^2 + (r_1\sin\theta_1 - r_2\sin\theta_2)^2}$

Proof Sketch: Variational formulas

Formulas for $\frac{d}{dt}(\lambda_1(t))|_{t=0}$ & $\frac{d}{dt}(\lambda_2(t))|_{t=0}$ come from analytic perturbation theory

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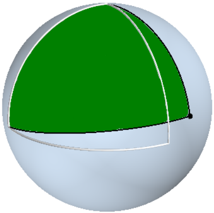

$$\begin{cases} \Delta_{\mathbb{S}^2}\phi(t) = -\lambda(t)\phi(t) & \text{in } T_t \\ \phi(t) = 0 & \text{on } \partial T_t \end{cases} \Rightarrow \begin{cases} \Delta_t\phi(t) = -\lambda(t)\phi(t) & \text{in } T_0 \\ \phi(t) = 0 & \text{on } \partial T_0 \end{cases}$$

See Seto, Wei, and Zhu, "Fundamental Gaps of Spherical Triangles"

Proof Sketch: Variational formula for λ_1

Lemma (El Soufi and Ilias)

$$\frac{d}{dt}\lambda_1(t)|_{t=0} = -\int_{T_0} \phi_1(0)\Delta'\phi_1(0)v_g$$

- $\Delta' = \frac{d}{dt}\Delta_{g_t}|_{t=0}$
- $\phi_1(0)$ *is the normalized eigenfunction for* λ_1 *on* T_0

Proof Sketch: Variational formula for λ_1

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Seto, Wei, and Zhu compute

- $\phi_1(0) = \sqrt{\frac{105}{2\pi}}\sin^2(\theta)\cos(\theta)\sin(2\varphi)$
- an explicit formula for Δ'

Lemma (Seto, Wei, and Zhu)

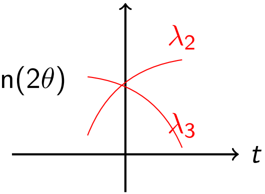
$$\frac{d}{dt}\lambda_1(t)|_{t=0} = -\int_{T_0} \phi_1(0)\Delta'\phi_1(0)v_g = \underbrace{\ldots\ldots\ldots\ldots}_{\text{messy computations}} = -\frac{28}{\pi}(b+a)$$

Proof Sketch: Variational formula for λ_2

There are two eigenfunctions corr. to λ_2 of T_0:

$$\phi_2^{(1)}(0) = \sqrt{\frac{1155}{8\pi}}(3\cos^5(r) - 4\cos^3(r) + \cos(r))\sin(2\theta)$$

$$\phi_2^{(2)}(0) = \sqrt{\frac{3465}{32\pi}}\cos(r)\sin^4(r)\sin(4\theta)$$



Lemma (El Soufi and Ilias)

There exist $(\lambda_2^{(1)}(t), \phi_2^{(1)}(t))$, $(\lambda_2^{(2)}(t), \phi_2^{(2)}(t))$ *such that*
$\lambda_2^{(i)}(0) = 30$
$\frac{d}{dt}(\lambda_2^{(i)}(t))|_{t=0}$ *are eigenvalues of the quadratic form*

$$\phi \to -\int_{T_0} \phi \Delta' \phi v_g$$

Proof Sketch: Variational formula for λ_2

The quadratic form has matrix:

$$\begin{pmatrix} b\dfrac{77}{\pi} + a\dfrac{44}{\pi} & -b\dfrac{22\sqrt{3}}{\pi} \\ -b\dfrac{22\sqrt{3}}{\pi} & b\dfrac{55}{\pi} + a\dfrac{88}{\pi} \end{pmatrix}$$

This quadratic form is nondegenerate when $a = -b$.

Theorem (Hillairet, J., Raschel)

There exists a one-parameter family of spherical triangles such that

- $\lambda_1(t) = 12 \left(\Rightarrow \alpha_1 = \dfrac{9}{2} \right)$
- $\lambda_2(t)$ *is real-analytic and non-constant.* ($\Rightarrow \alpha_p$ *nonconstant*)

Thank you for listening!

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