Combinatorics of the dP_3 Quiver

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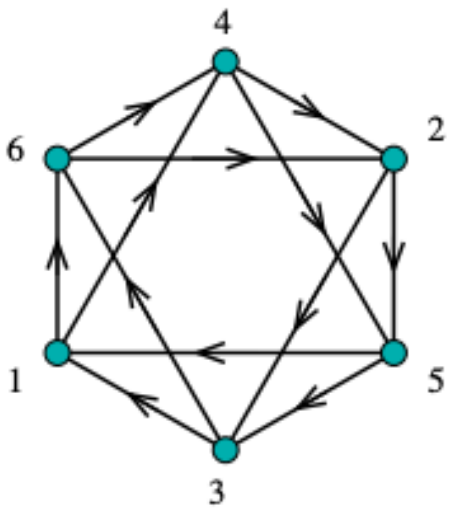
Joint work with Tri Lai and Gregg Musiker

Introduction

Object of study. The dP_3 *quiver*[1] and its associated *cluster algebra*.

Goal. Provide combinatorial interpretations for *toric cluster variables* obtained from sequences of *mutations*.



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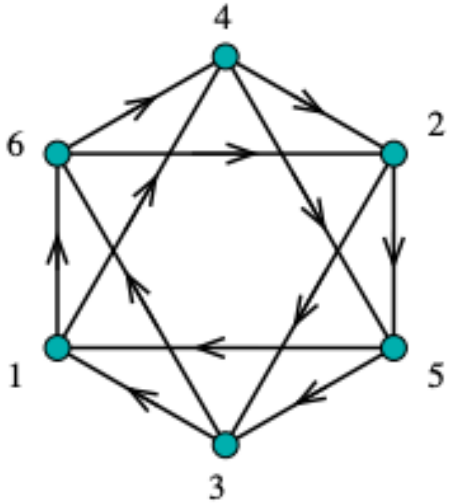

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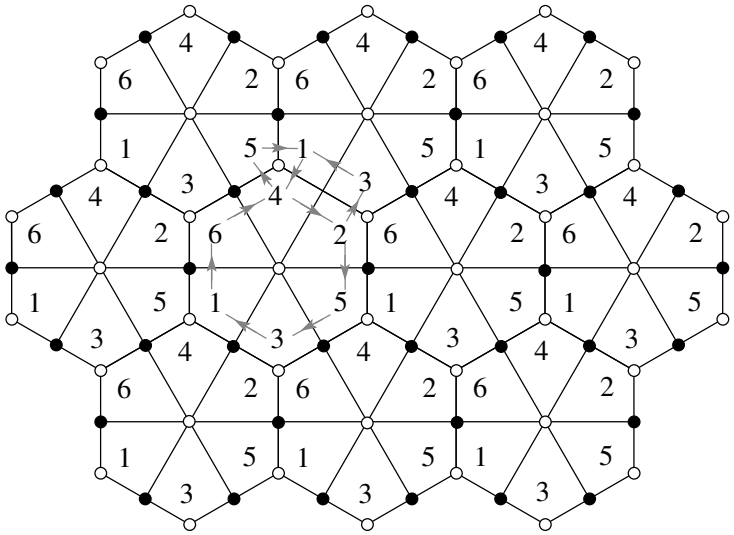


Previous work. [Z12, LMNT, LM17, LM20] In most cases, toric CVs can be interpreted using the *dimer model*.

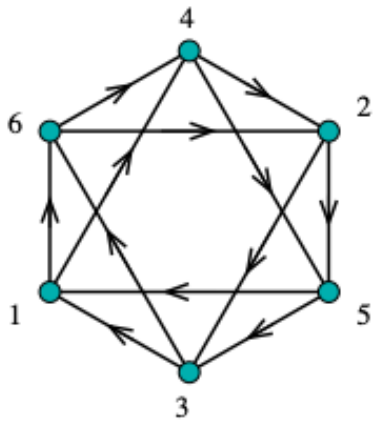
Current work. In the remaining cases, we give a combinatorial interpretation using the *tripartite double-dimer model*.

[1] The quiver Q associated with the Calabi-Yau threefold complex cone over the third del Pezzo surface of degree 6 ($\mathbb{CP}^2$ blown up at three points).

Images shown are Figures 1 and 2 from T. Lai and G. Musiker, *Dungeons and Dragons: Combinatorics for the dP_3 Quiver*

Quiver, quiver mutations, and cluster variables

A *quiver* Q is a directed finite graph.



Q

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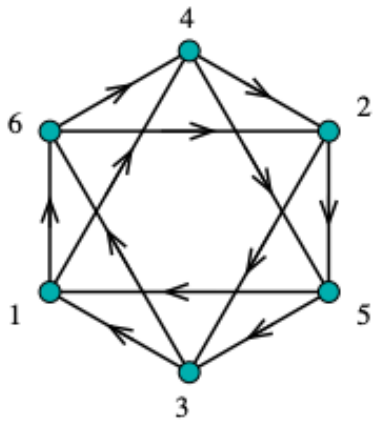
Definition (Mutation at a vertex i)

- For every 2-path $j \to i \to k$, add $j \to k$
- Reverse all arrows incident to i
- Delete 2-cycles



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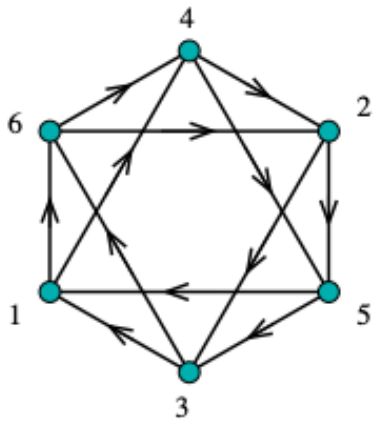

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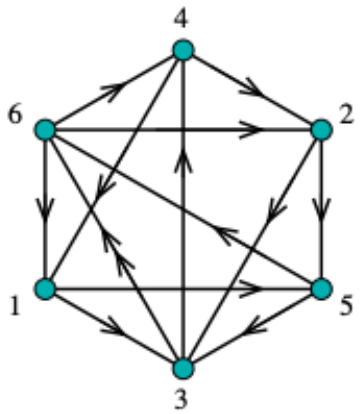

$\mu_1(Q)$

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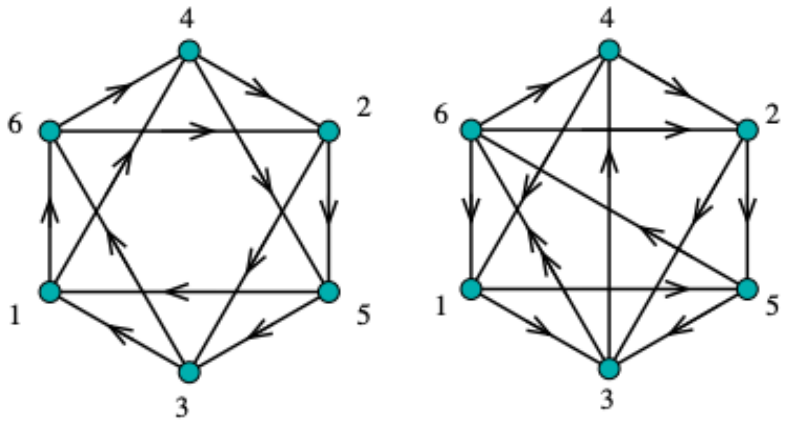


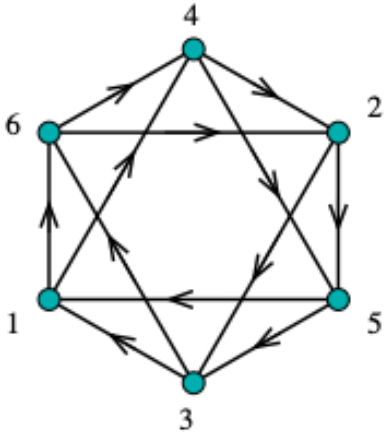
Q $\qquad$ $\mu_1(Q)$

- Define a cluster algebra from a quiver Q by associating a cluster variable x_i to every vertex labeled i.
- When we mutate at vertex i we replace x_i with x_i', where

$$x_i' = \frac{\prod_{i \to j \text{ in } Q} x_j^{a_{i\to j}} + \prod_{j \to i \text{ in } Q} x_j^{b_{j\to i}}}{x_i}$$

- When we mutate at 1 we replace x_1 with $x_1' = \dfrac{x_4x_6 + x_3x_5}{x_1}$. Now we have the cluster: $\left\{\frac{x_4x_6+x_3x_5}{x_1}, x_2, x_3, \ldots, x_6\right\}$

Quiver, quiver mutations, and cluster variables



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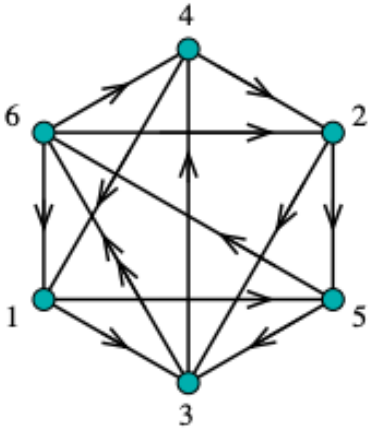

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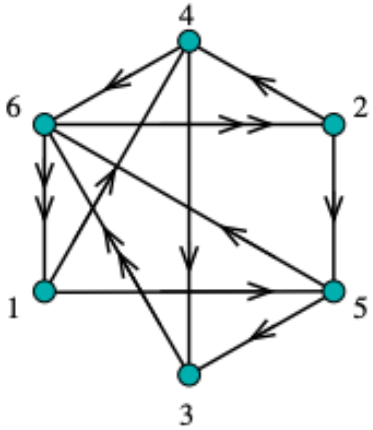

$\mu_1\mu_4(Q)$

Mutate at 4: replace x_4 with

$$x_4' = \frac{x_3x_6 + x_2x_1'}{x_4} = \frac{x_1x_3x_6 + x_2x_3x_5 + x_2x_4x_6}{x_1x_4}$$

Now we have the cluster: $\left\{\frac{x_4x_6+x_3x_5}{x_1}, x_2, x_3, \frac{x_1x_3x_6+x_2x_3x_5+x_2x_4x_6}{x_1x_4}, x_5, x_6\right\}$

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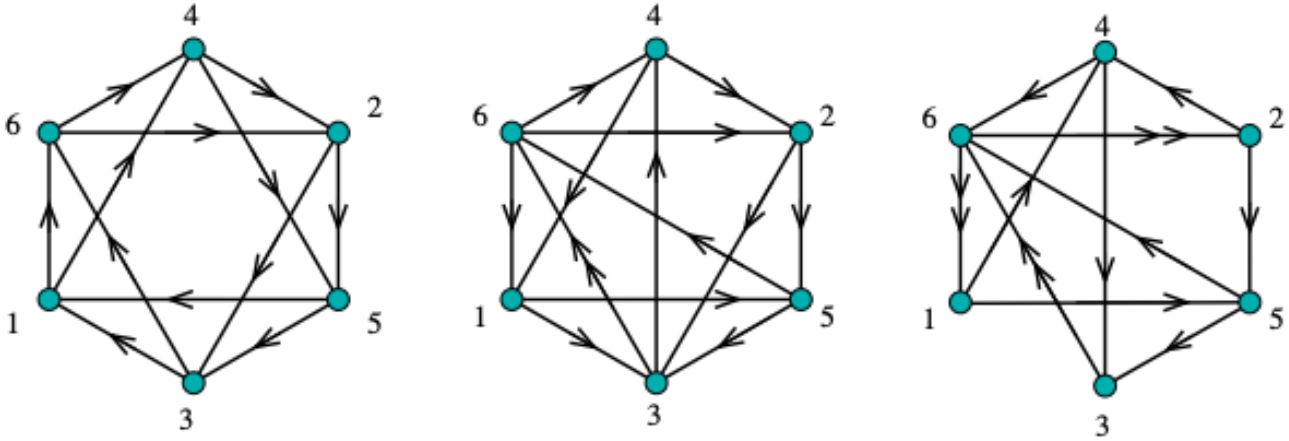

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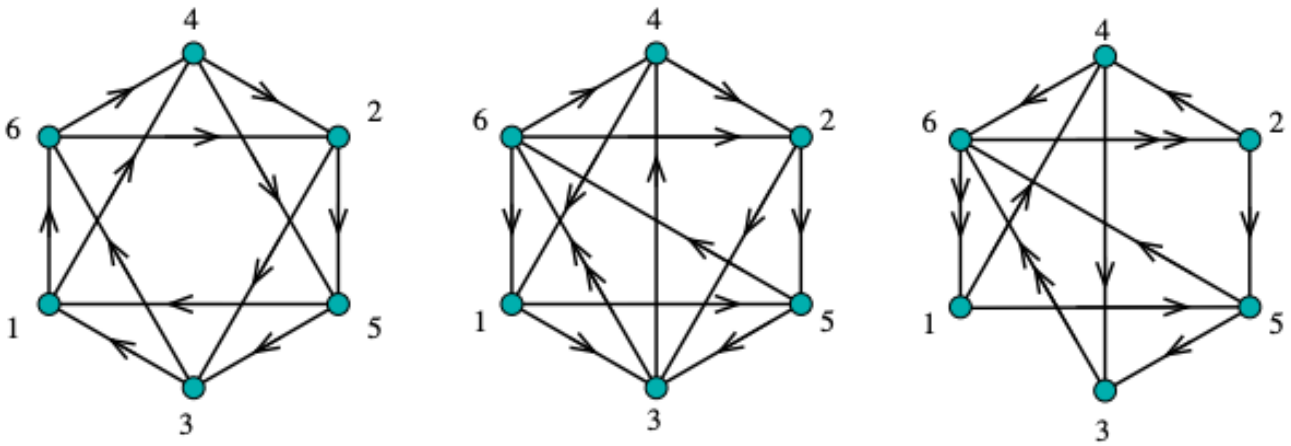
Theorem (FZ02)

Every cluster variable is a Laurent polynomial in $x_1, \ldots, x_n$.

Image shown is Figure 2 from T. Lai and G. Musiker, *Dungeons and Dragons: Combinatorics for the dP_3 Quiver*

Quiver, quiver mutations, and cluster variables



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- A *toric mutation* is a mutation at a vertex with both in-degree and out-degree 2.

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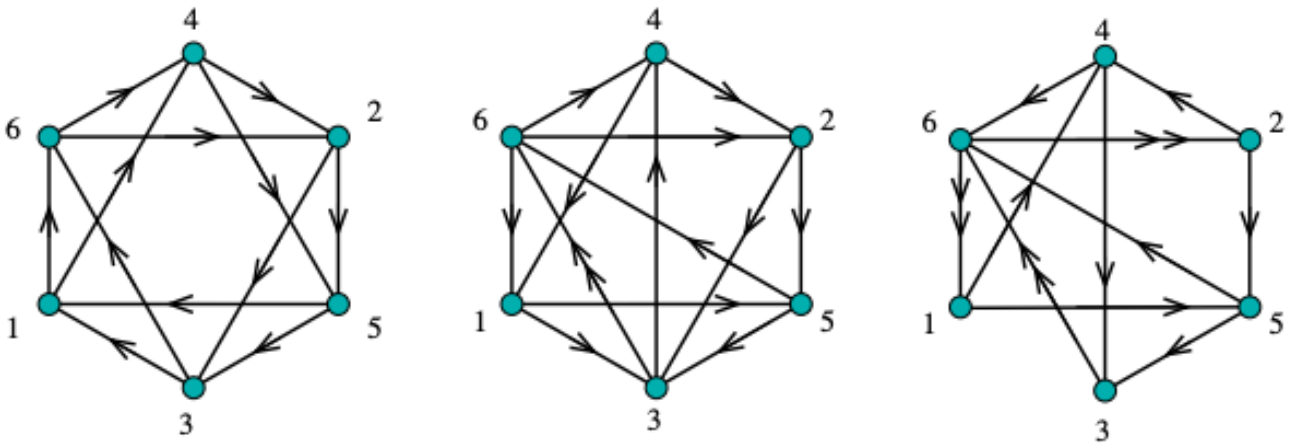

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- A *toric mutation* is a mutation at a vertex with both in-degree and out-degree 2.
- A *toric cluster variable* is a cluster variable arising from a sequence of toric mutations.

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$\mathbb{Z}^3$ parameterization for toric cluster variables and an algebraic formula

Lai and Musiker (LM17) showed that for the dP_3 quiver, the set of toric cluster variables is parameterized by $\mathbb{Z}^3$.

Let $z_{i,j,k}$ denote the toric cluster variable corresponding to $(i,j,k) \in \mathbb{Z}^3$.

Theorem (LM17)

Let $A = \frac{x_3x_5+x_4x_6}{x_1x_2}$, $B = \frac{x_1x_6+x_2x_5}{x_3x_4}$, $C = \frac{x_1x_3+x_2x_4}{x_5x_6}$, $D = \frac{x_1x_3x_6+x_2x_3x_5+x_2x_4x_6}{x_1x_4x_5}$, $E = \frac{x_2x_4x_5+x_1x_3x_5+x_1x_4x_6}{x_2x_3x_6}$. *Then*

$$z_{i,j,k} = x_r A^{\lfloor \frac{(i^2+ij+j^2+1)+i+2j}{3} \rfloor} B^{\lfloor \frac{(i^2+ij+j^2+1)+2i+j}{3} \rfloor} C^{\lfloor \frac{i^2+ij+j^2+1}{3} \rfloor} D^{\lfloor \frac{(k-1)^2}{4} \rfloor} E^{\lfloor \frac{k^2}{4} \rfloor}$$

x_r is an initial cluster variable with r depending on $(i-j)$ mod 3 and k mod 2.

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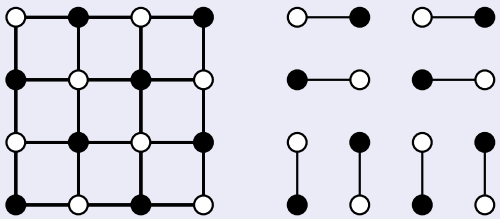
x_r *is an initial cluster variable with* r *depending on* $(i-j)$ *mod* 3 *and* k *mod* 2.

In most cases, these algebraic formulas agree with the generating function for dimer configurations of certain graphs!

Dimer configurations

Assume we have a finite, bipartite, planar graph.

Definition (Dimer configuration/Perfect matching)



A collection of edges that covers each vertex exactly once

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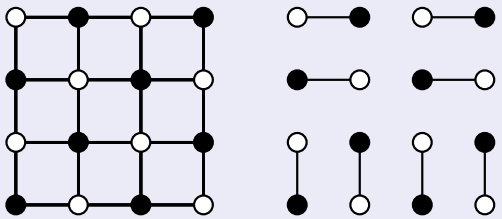

A collection of edges that covers each vertex exactly once

- Given a graph, we can assign a weight $w(e)$ to each edge.
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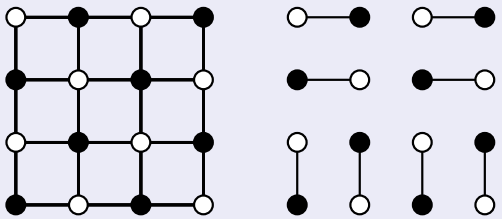

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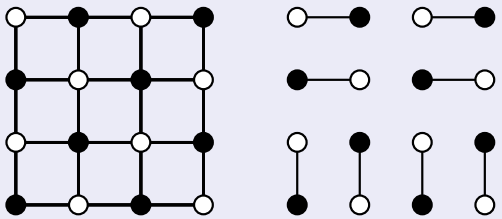

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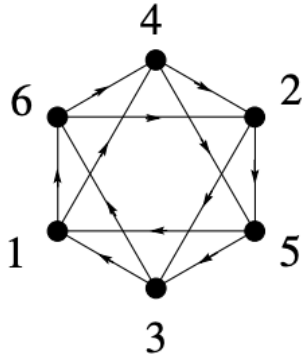
- Let $Z^D(G) = \sum_M w(M)$, called the *partition function*.
- The algebraic formulas from LM17 are counting dimer configurations of certain subgraphs of the *brane tiling* associated to the dP_3 quiver.

The dP_3 quiver and its brane tiling

- A *brane tiling* is a doubly periodic bipartite planar graph that can be associated to a pair (Q, W), where W is a *potential*.



$$
\begin{aligned}
W \;=\; & A_{16}A_{64}A_{42}A_{25}A_{53}A_{31} && + A_{14}A_{45}A_{51} \\
& +A_{23}A_{36}A_{62} && - A_{16}A_{62}A_{25}A_{51} \\
& -A_{36}A_{64}A_{45}A_{53} && - A_{14}A_{42}A_{23}A_{31}
\end{aligned}
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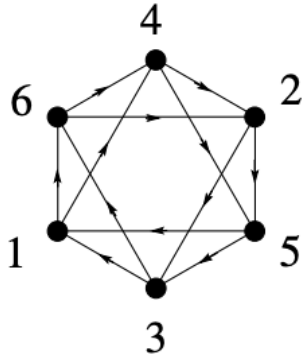
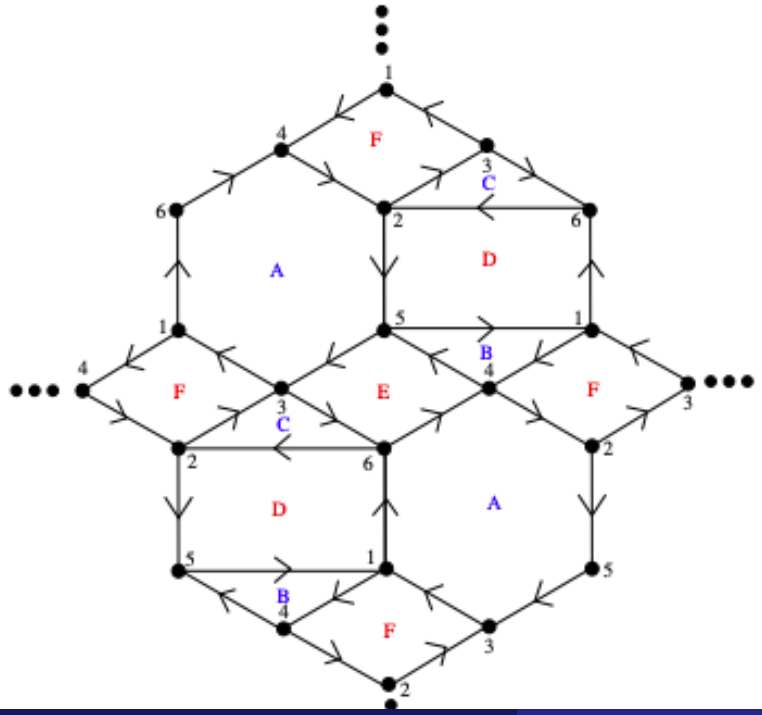
W \;=\; & A_{16}A_{64}A_{42}A_{25}A_{53}A_{31}(A) + A_{14}A_{45}A_{51}(B) \\
& + A_{23}A_{36}A_{62}(C) - A_{16}A_{62}A_{25}A_{51}(D) \\
& - A_{36}A_{64}A_{45}A_{53}(E) - A_{14}A_{42}A_{23}A_{31}(F)
\end{aligned}
$$



Unfold Q onto a planar directed graph $\tilde{Q}$



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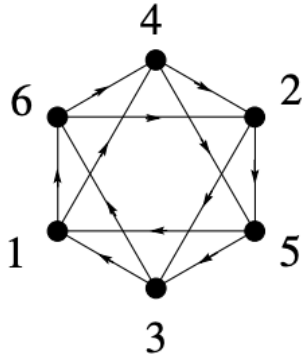
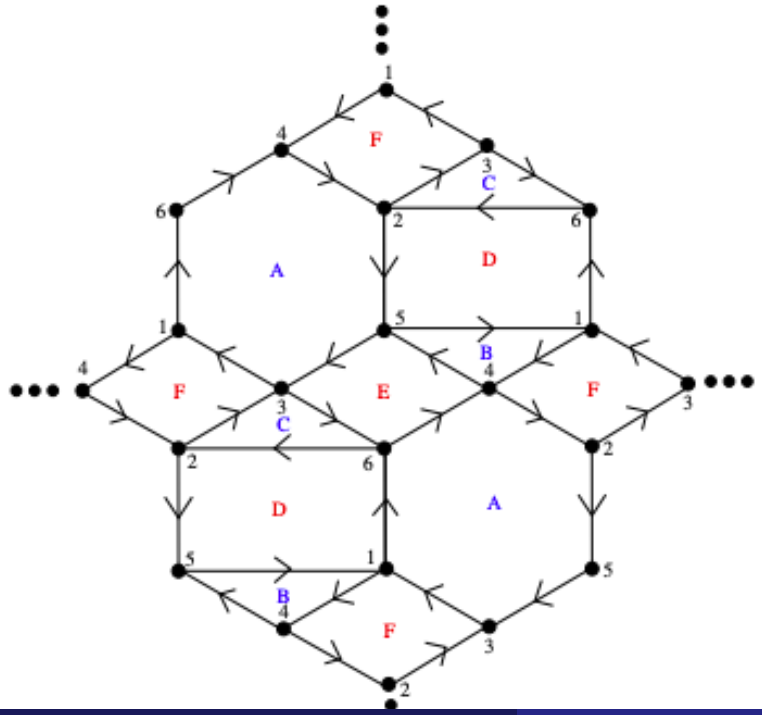
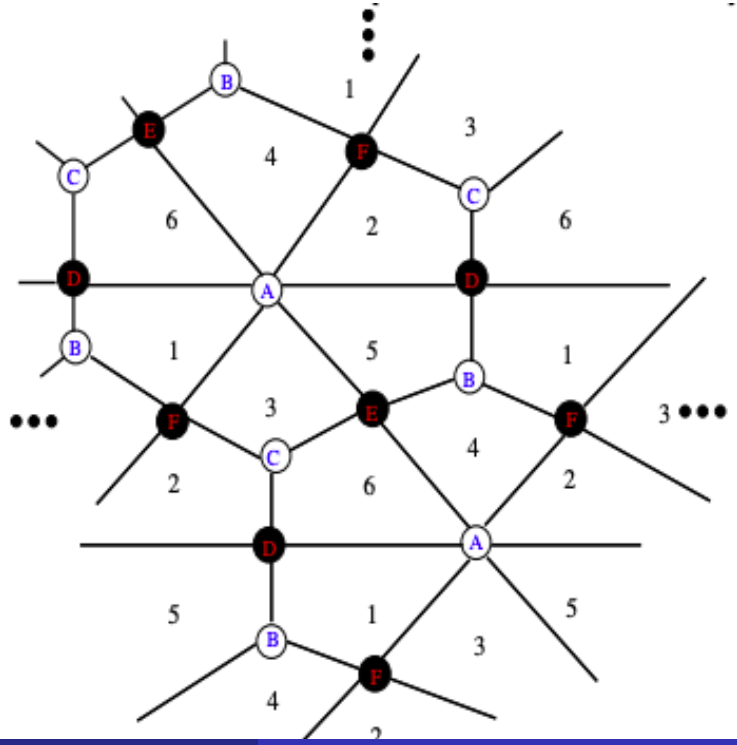


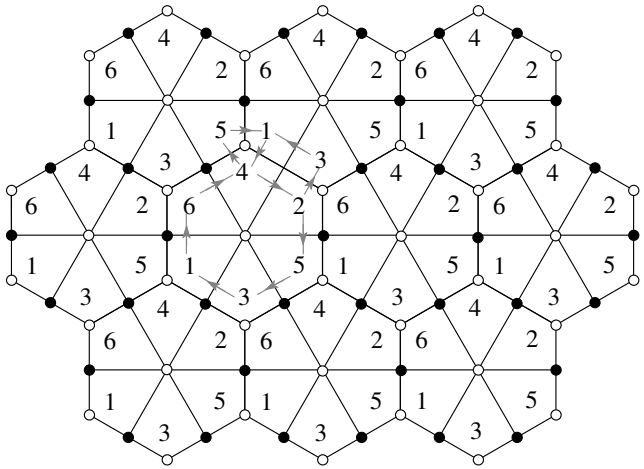
Unfold Q onto a planar directed graph $\tilde{Q}$, then take the dual:



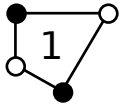


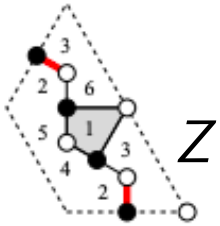
Combinatorial interpretation: Example 1

- The edge bordering faces i and j gets weight $\frac{1}{x_i x_j}$
- Define the covering monomial $m(G) = \prod_{i=1}^{6} x_i^{a_i}$, where $a_i = \#$ faces labeled i enclosed in or bordering G.



Example. After mutating Q at vertex 1 we got $x_1' = \frac{x_4x_6+x_3x_5}{x_1}$

Let G be the graph 

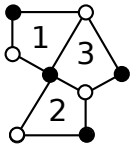


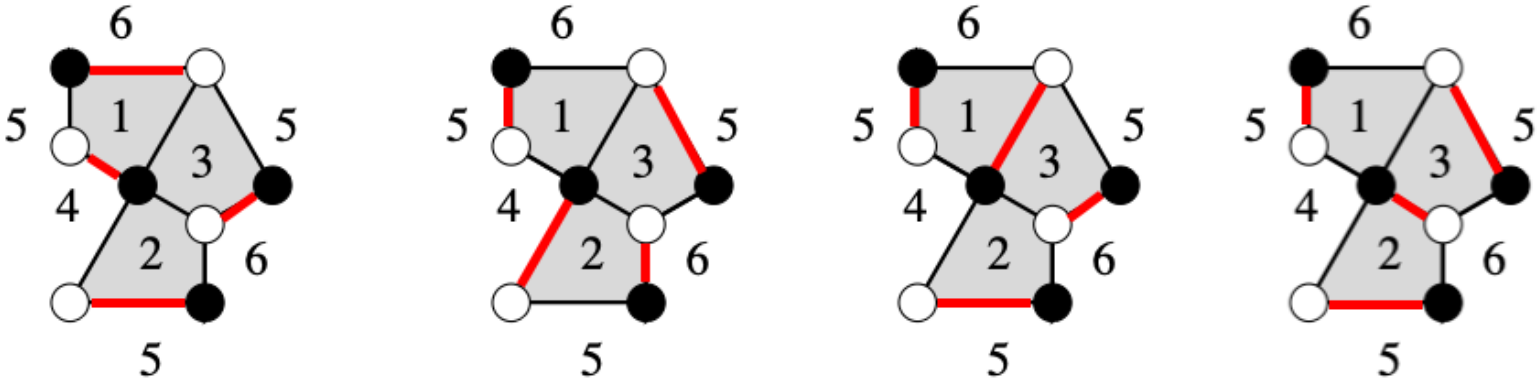
$$Z^D(G)m(G) = \left(\frac{1}{x_1x_3}\cdot\frac{1}{x_1x_5} + \frac{1}{x_1x_4}\cdot\frac{1}{x_1x_6}\right)x_1x_3x_4x_5x_6 = \frac{x_4x_6+x_3x_5}{x_1}$$

Combinatorial interpretation: Example 2

Example.

x_3 in $\mu_1\mu_2\mu_3(Q)$: $\dfrac{x_2x_3x_5^2+x_1x_3x_5x_6+x_2x_4x_5x_6+x_1x_4x_6^2}{x_1x_2x_3}$

Let G be the graph 



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Constructing subgraphs of the brane tiling

Map from $\mathbb{Z}^3$ to $\mathbb{Z}^6$:

$$(i, j, k) \to (a, b, c, d, e, f) = (j + k, -i - j - k, i + k, j - k + 1, -i - j + k - 1, i - k + 1)$$

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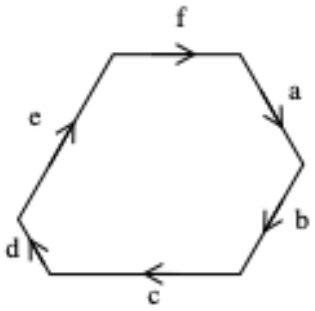
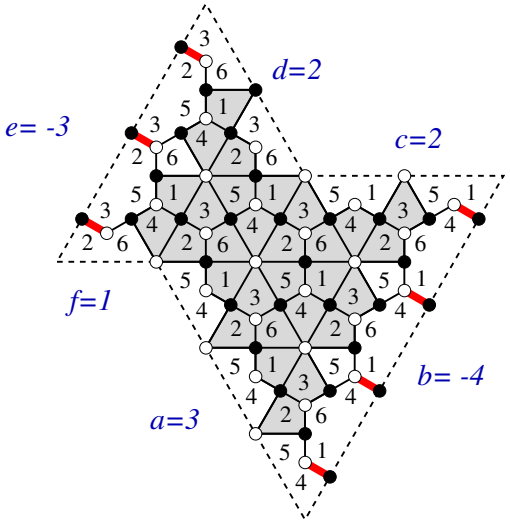
$$(i,j,k) \to (a,b,c,d,e,f) = (j+k, -i-j-k, i+k, j-k+1, -i-j+k-1, i-k+1)$$

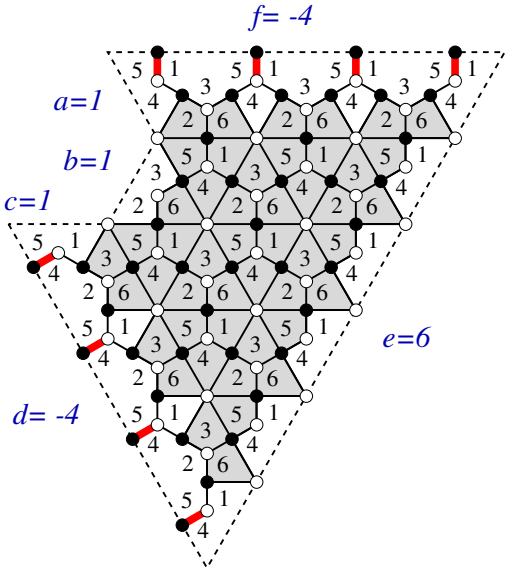
Given a six-tuple $(a,b,c,d,e,f) \in \mathbb{Z}^6$, superimpose the contour $C(a,b,c,d,e,f)$ on the dP_3 lattice.
Magnitude determines length and sign determines direction.

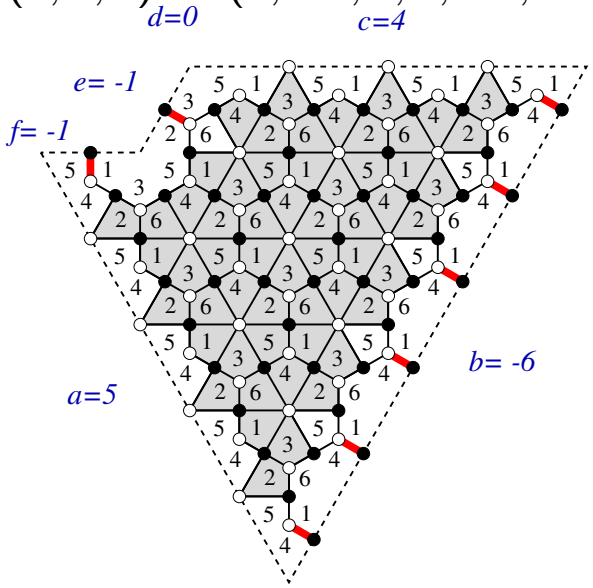


Examples:

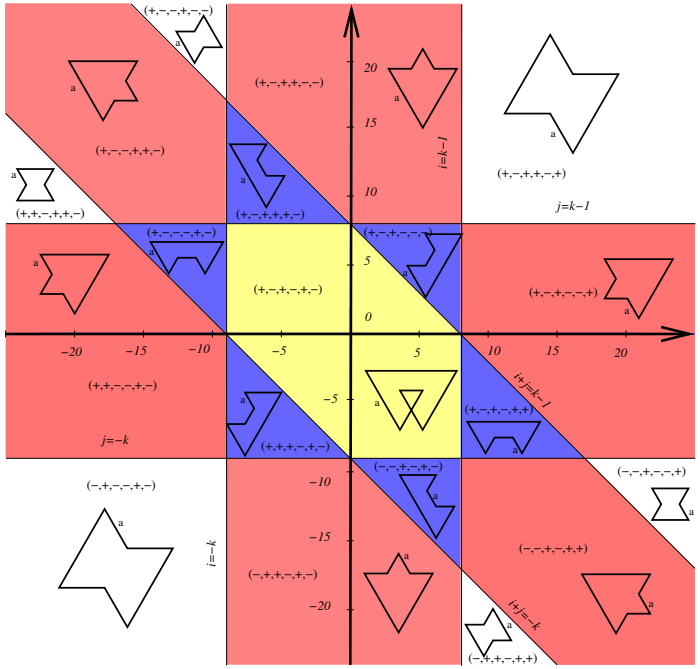
$(1,2,1) \to (3,-4,2,2,-3,1)$ $(-2,-2,3) \to (1,1,1,-4,6,-4)$ $(1,2,3) \to (5,-6,4,0,-1,-1)$







Combinatorial interpretation of $z_{i,j,k}$



Possible shapes of the contours for a fixed $k \geq 1$

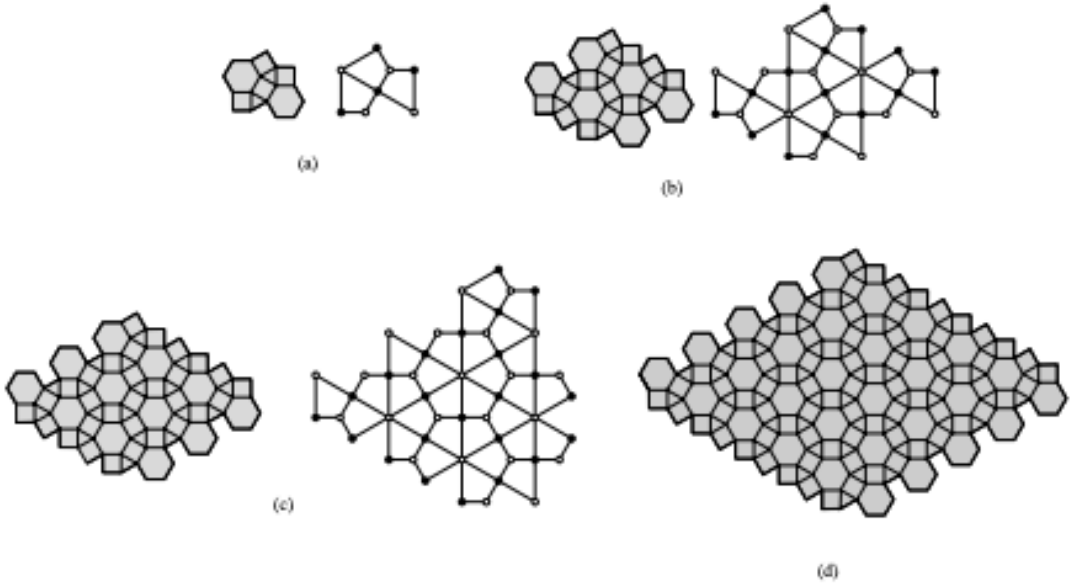
Theorem (LM17)

Let G be the subgraph cut out by the contour
$(a, b, c, d, e, f) = (j + k, -i - j - k, i + k, j - k + 1, -i - j + k - 1, i - k + 1)$.
As long as $C(a, b, c, d, e, f)$ has no self-intersections, $z_{i,j,k} = Z^D(G)m(G)$

Image shown is Figure 20 from T. Lai and G. Musiker, *Beyond Aztec Castles: Toric cascades in the dP_3 Quiver*

Aztec Dragons

In 1999, Propp's *Enumerations of matchings: Problems and Progress* contained a list of open problems that included proving enumeration formulas for several analogues of the Aztec Diamond.



Theorem (Wieland, Ciucu)

The number of tilings of the Aztec dragon of order n is $2^{n(n+1)}$.

Images shown are Figures 24 from T. Lai and G. Musiker, *Dungeons and Dragons: Combinatorics for the dP_3 Quiver* and Figure 14 from LM, *Beyond Aztec Castles*

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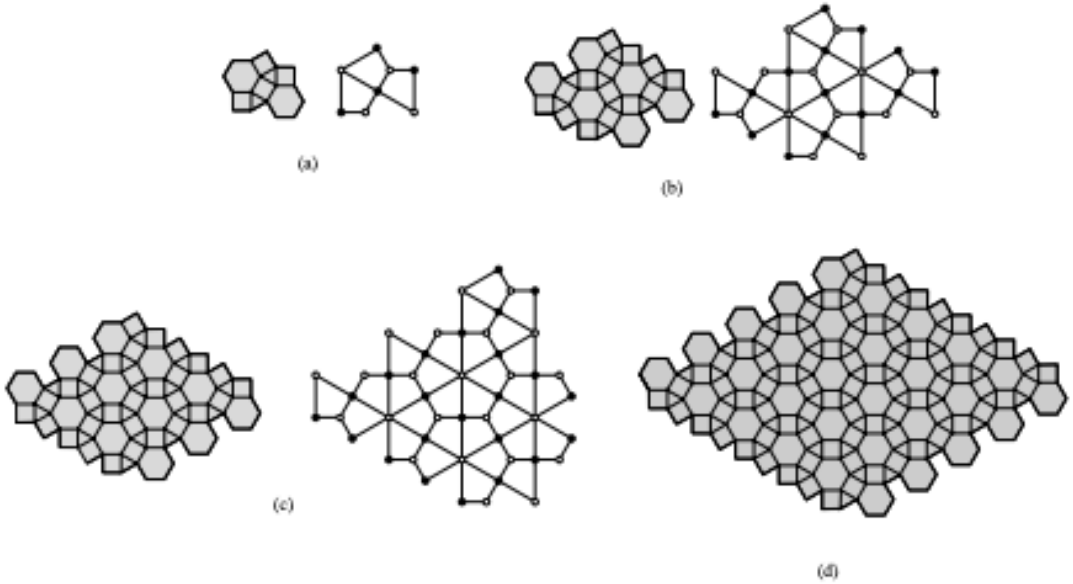

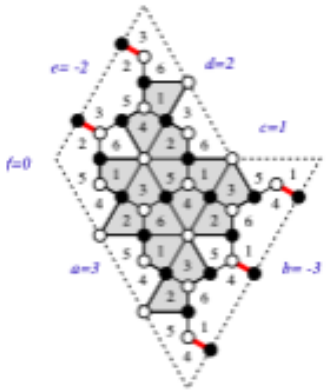

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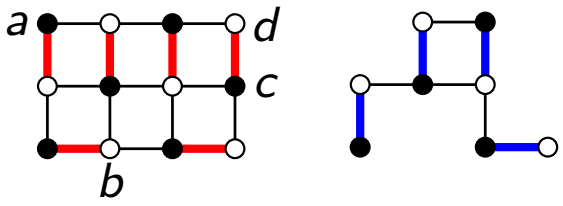
Kuo condensation

Theorem (Kuo04, Theorem 5.1)

Let vertices a, b, c, and d appear in a cyclic order on a face of G. If $a, c \in V_1$ and $b, d \in V_2$, then

$$Z^D(G)Z^D(G-\{a,b,c,d\})=Z^D(G-\{a,b\})Z^D(G-\{c,d\})+Z^D(G-\{a,d\})Z^D(G-\{b,c\})$$



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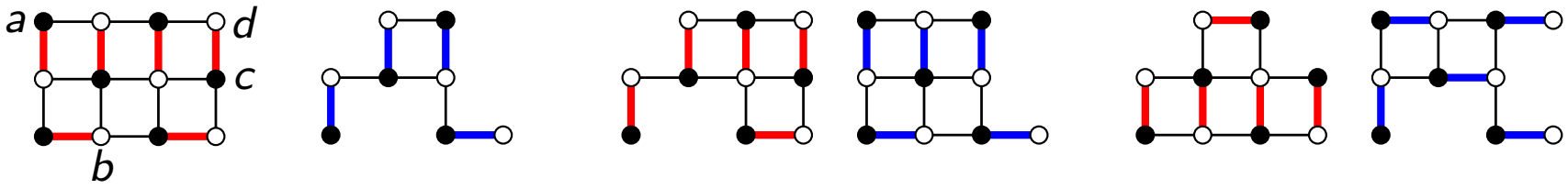

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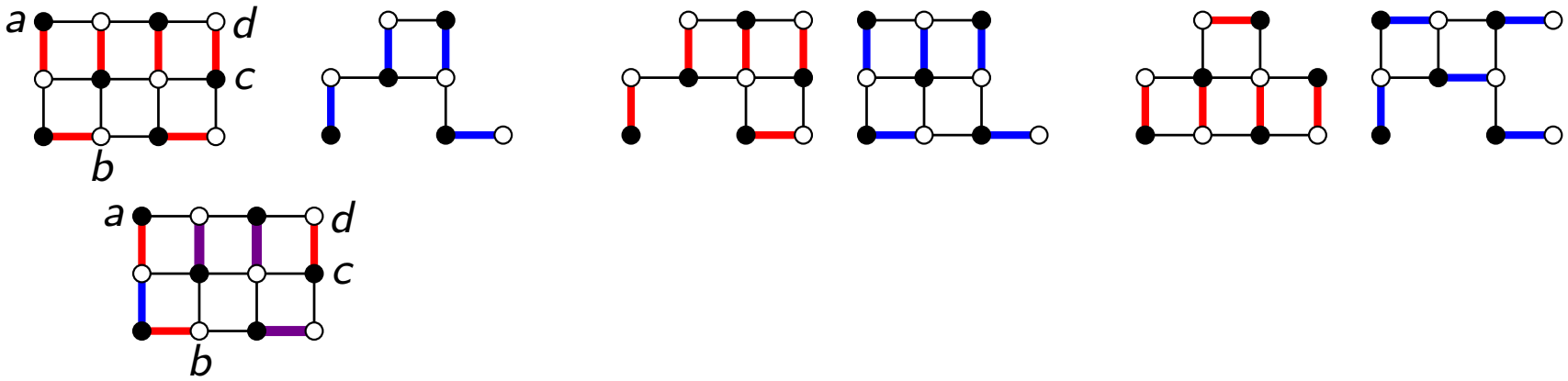

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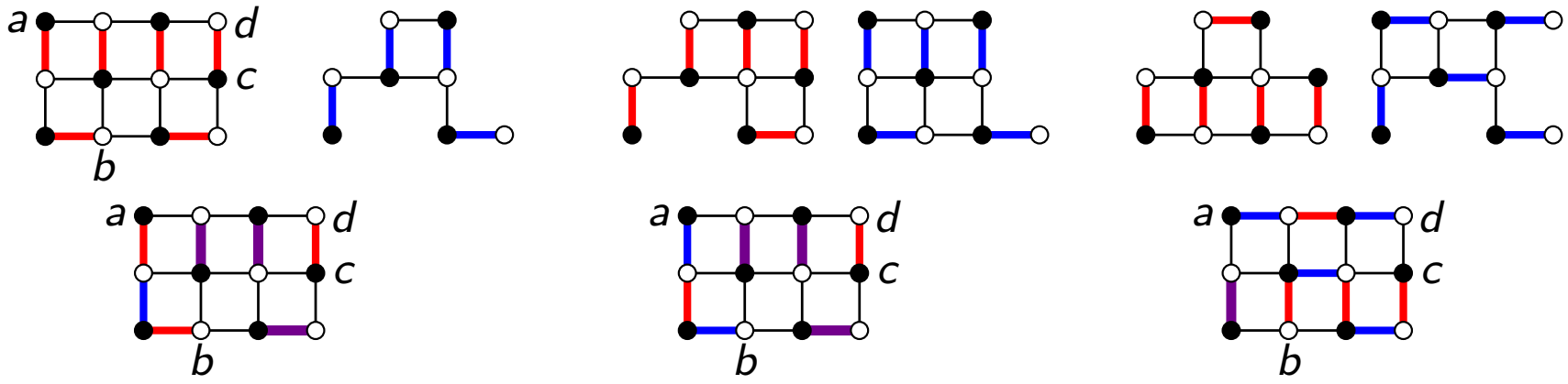

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Theorem (Desnanot-Jacobi identity/Dodgson condensation)

$$\det(M)\det(M_{i,j}^{i,j}) = \det(M_i^i)\det(M_j^j) - \det(M_i^j)\det(M_j^i)$$

M_i^j is the matrix M with the ith row and the jth column removed.

Proof of combinatorial interpretation

Theorem (LM17)

If G is a subgraph cut out by a contour with no self-intersections,

$$z_{i,j,k} = Z^D(G)m(G)$$

Idea: compare cluster mutations of $z_{i,j,k}$'s to Kuo condensation identities.

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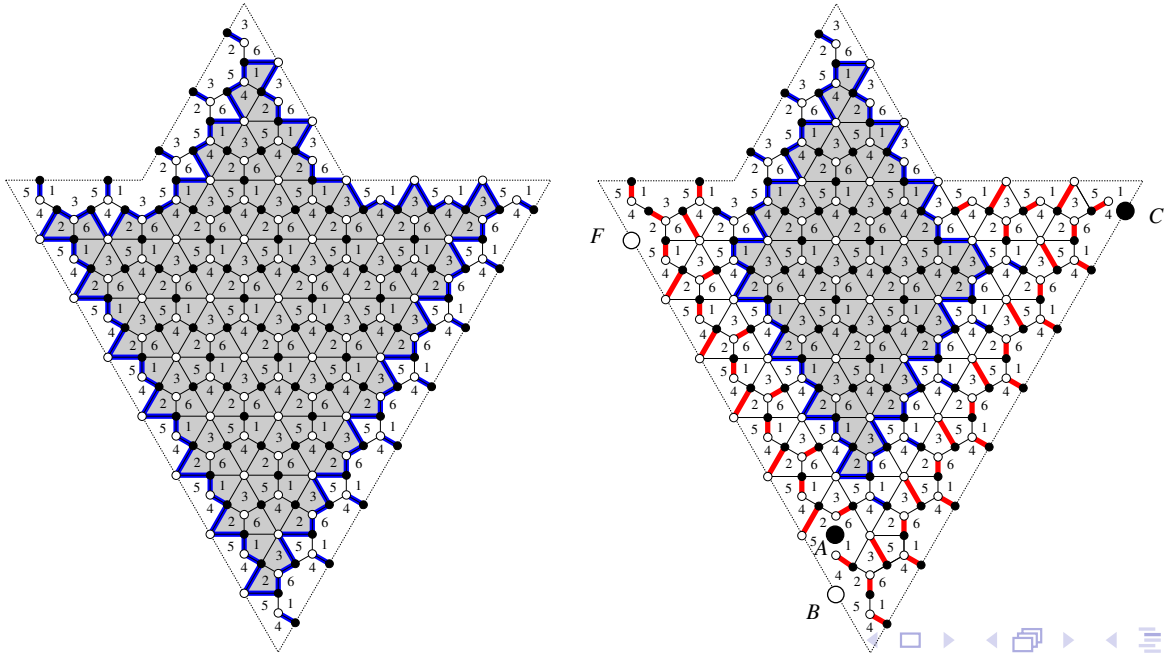

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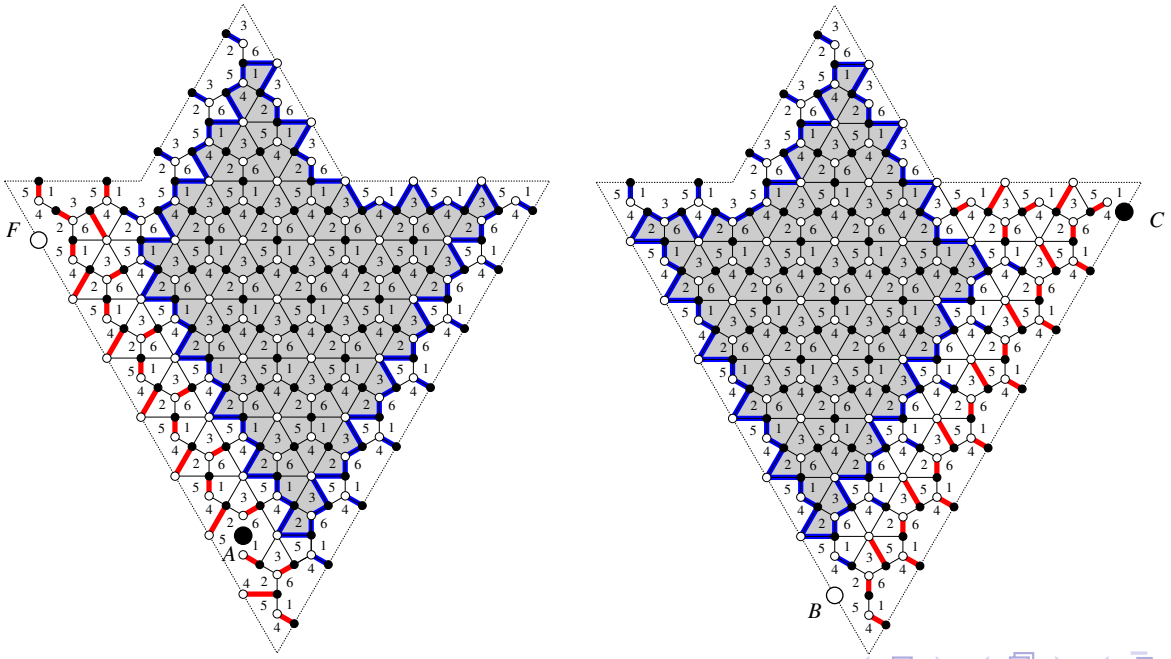

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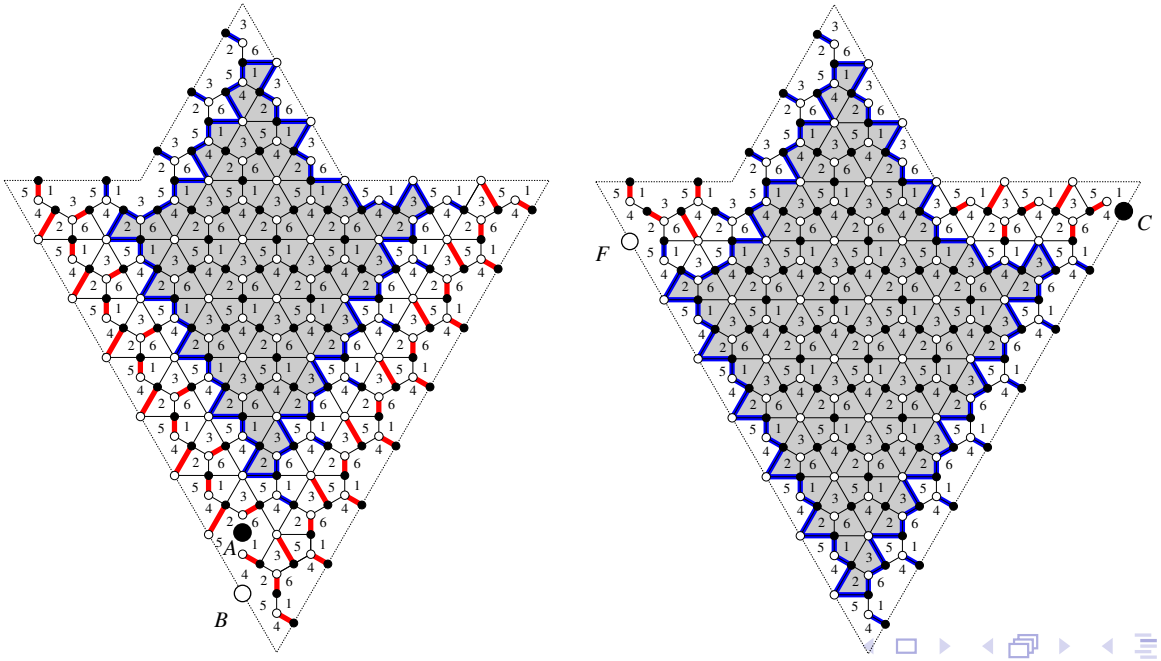
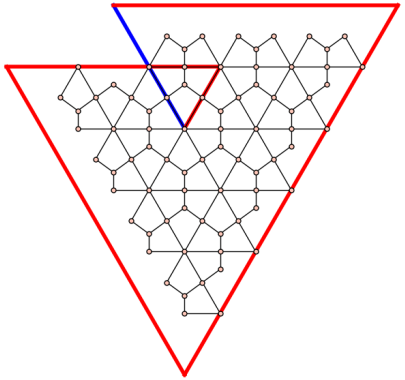


Self-intersecting contours

What about when the contour is self-intersecting?

The algebraic formula still holds, but the dimer interpretation does not.



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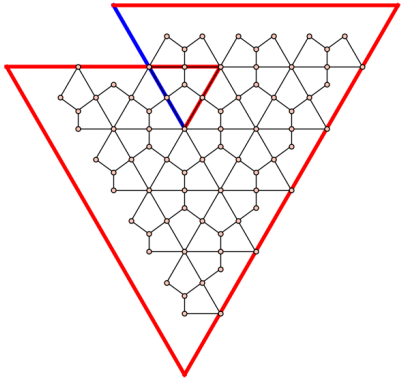
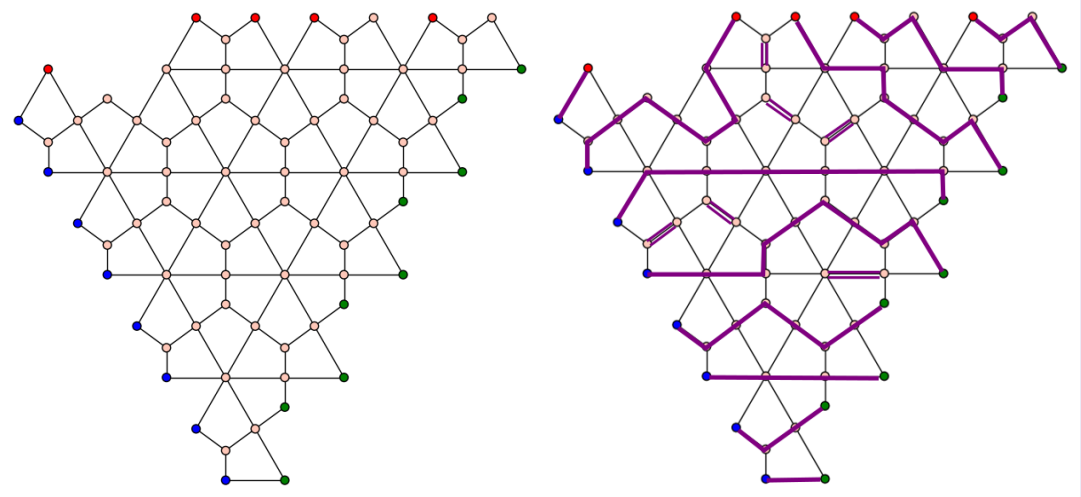


Definition (Double-dimer configuration on $(G, \mathbf{N})$)

Let $\mathbf{N}$ be a set of special vertices called *nodes* on the outer face of G.



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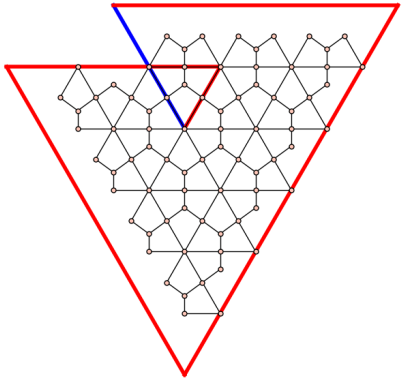

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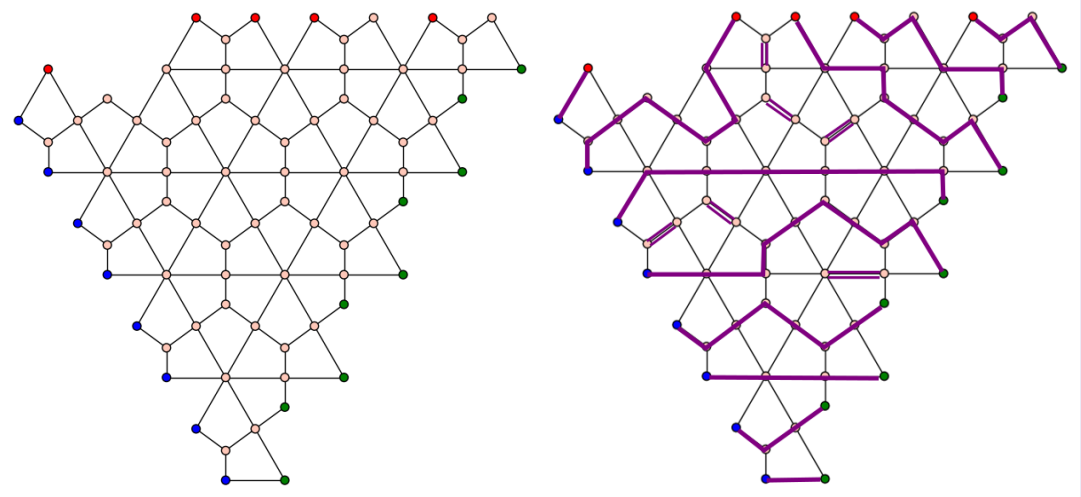

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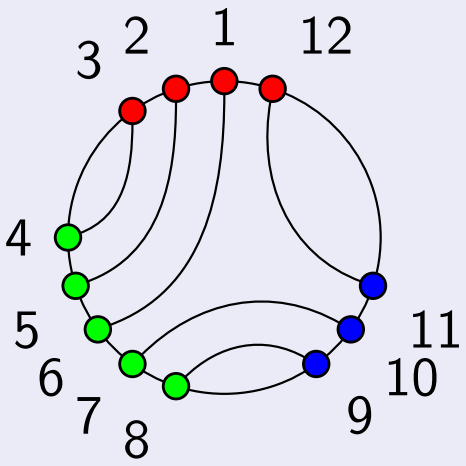
Weight is the product of edge weights $\times\ 2^{\ell}$

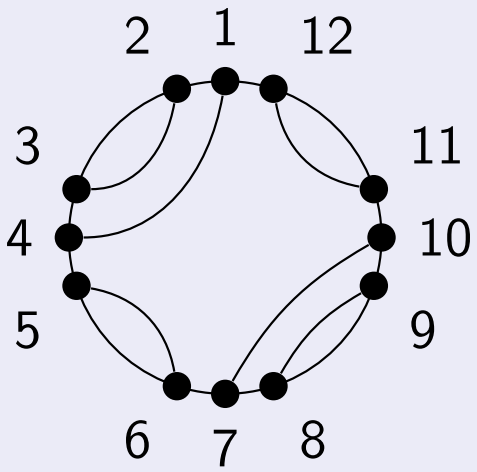
Tripartite pairings

Definition (Tripartite pairing)

A planar pairing σ of $\mathbf{N}$ is *tripartite* if the nodes can be divided into ≤ 3 sets of circularly consecutive nodes so that no node is paired with a node in the same set.



Tripartite



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We often color the nodes in the sets red, green, and blue, in which case σ has no monochromatic pairs.

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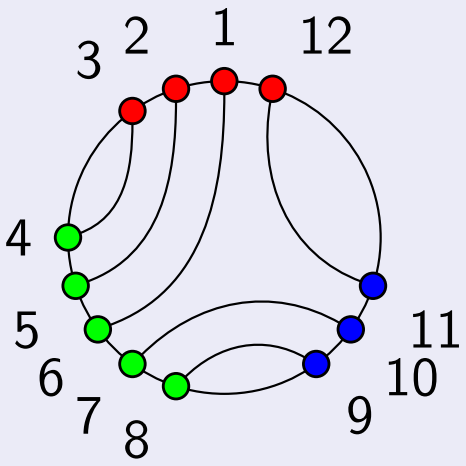

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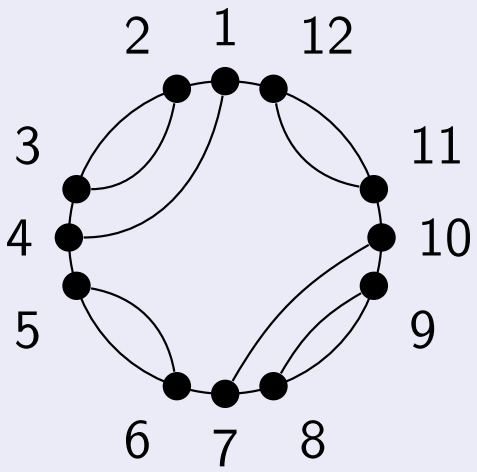

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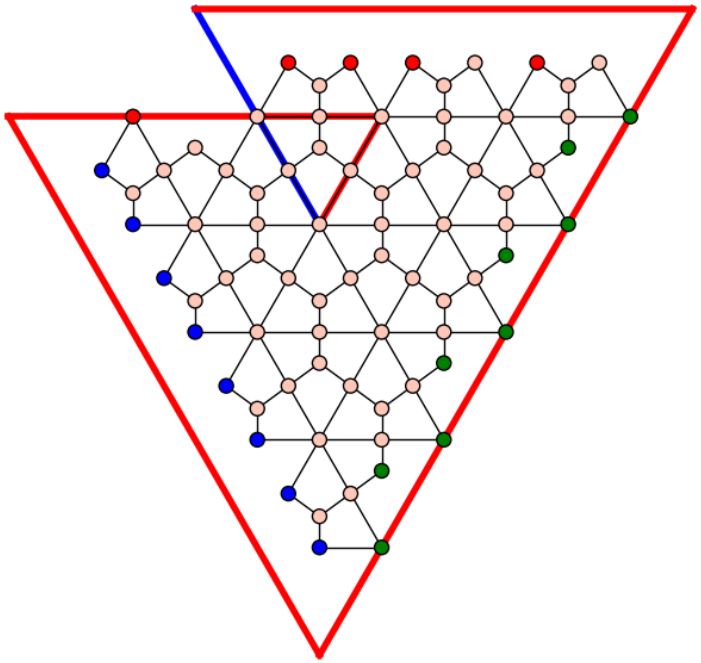
Dividing nodes into three sets R, G, and B defines a tripartite pairing.

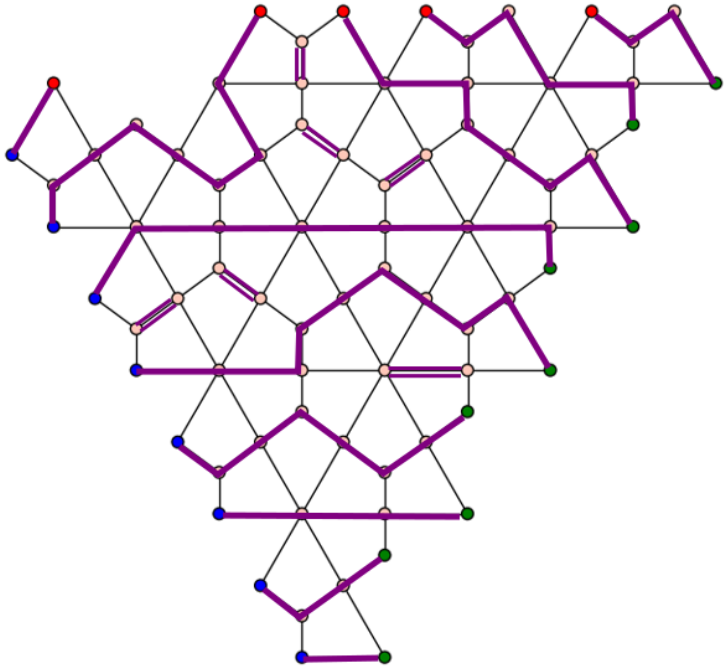
Combinatorial interpretation for self-intersecting contours

Theorem in progress (J-Lai-Musiker 2020+)

For the dP_3 quiver, we complete the assignment of combinatorial interpretations to toric cluster variables. In particular, for (i, j, k) associated to a self-intersecting contour we express $z_{i,j,k}$ as a partition function for a tripartite double-dimer configuration.

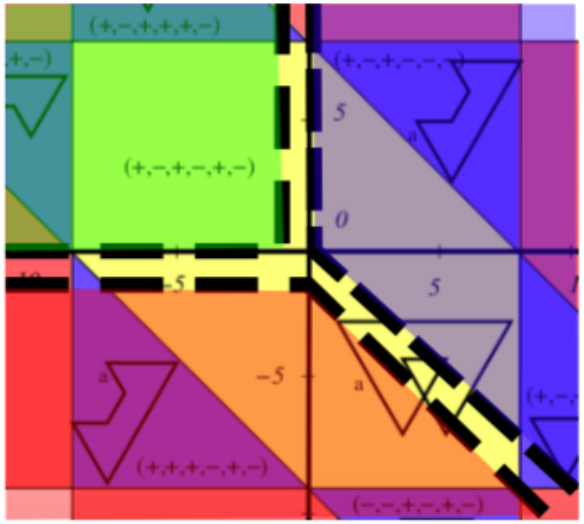


$z_{-1,-2,4}|_{x_i=1} = 11664$



There are 11664 tripartite DD configs

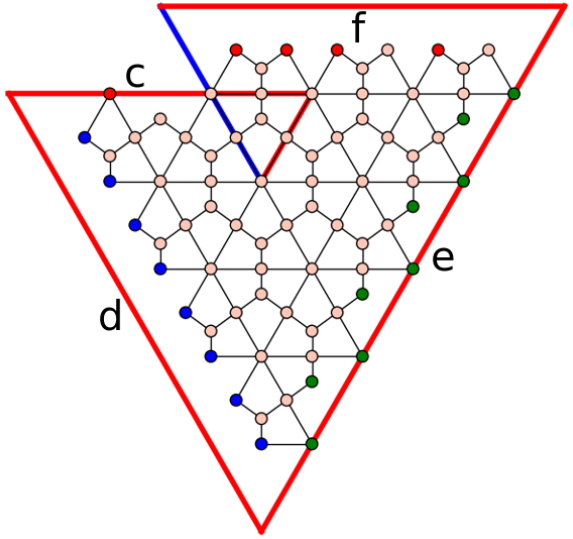
Description of node set



For fixed $k \geq 1$, we split the hexagon of lattice points corresponding to self-intersecting contours into three rhombi.

In the SW rhombic region, the nodes consist of

- All degree 2 vertices along edge d
- All degree 2 vertices along edge e
- Some degree 2 vertices along edges c and f



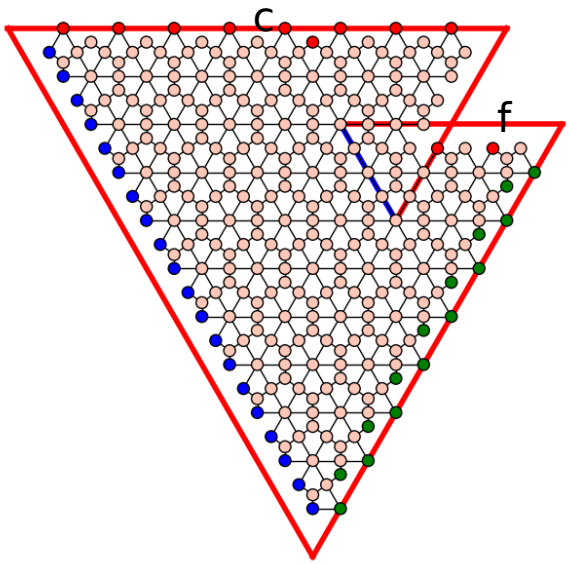
Description of node set

If $i \geq 0$, the red nodes are

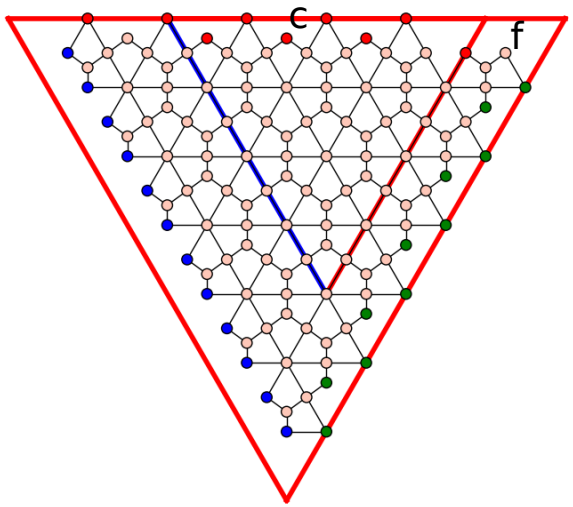
- Every other degree 2 vertex along edge f (until we reach the self-intersection)
- $|c| - 1$ "peaks" $+ (j + k - 1)$ "extra" vertices starting $i + 1$ peaks from the right

If $i < 0$, the red nodes are

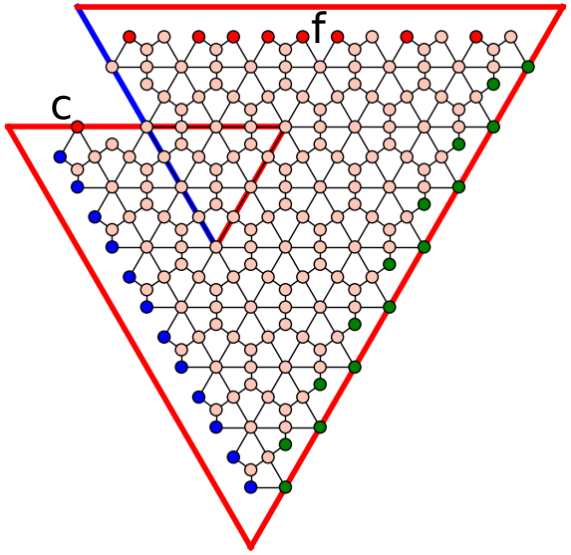
- Every other degree 2 vertex along edge c (until we reach the self-intersection)
- $|f| - 1$ left vertices $+ (i + j + k)$ "extra" vertices starting $-i$ from the left



$(2, -5, 7)$
$(2, -4, 9, -11, 9, -4)$



$(0, -2, 6)$
$(4, -4, 6, -7, 7, -5)$



$(-2, -2, 6)$
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Double-dimer condensation

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Theorem (J.)

Divide $\mathbf{N}$ *into sets* R, G, *and* B *and let* σ *be the corr. tripartite pairing. Let* $x, y, w, v \in \mathbf{N}$ *such that* $x < w \in V_1$ *and* $y < v \in V_2$. *If* $\{x, y, w, v\}$ *contains at least one node of each RGB color and* x, y, w, v *appear in cyclic order then*

$$Z_\sigma^{DD}(G, \mathbf{N}) Z_{\sigma_{xywv}}^{DD}(G, \mathbf{N} - \{x, y, w, v\}) =$$

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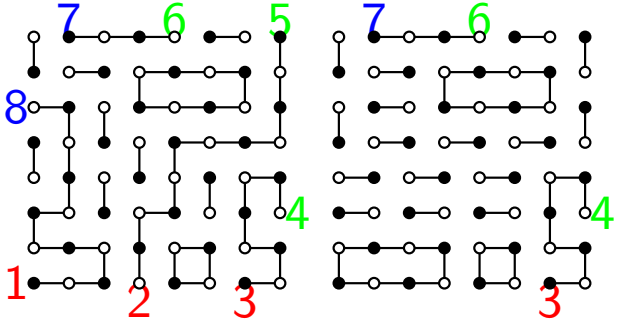
Example.

$$Z_\sigma^{DD}(\mathbf{N}) Z_{\sigma_{1258}}^{DD}(\mathbf{N} - 1, 2, 5, 8) = Z_{\sigma_{12}}^{DD}(\mathbf{N} - 1, 2) Z_{\sigma_{58}}^{DD}(\mathbf{N} - 5, 8) + Z_{\sigma_{18}}^{DD}(\mathbf{N} - 1, 8) Z_{\sigma_{25}}^{DD}(\mathbf{N} - 2, 5)$$



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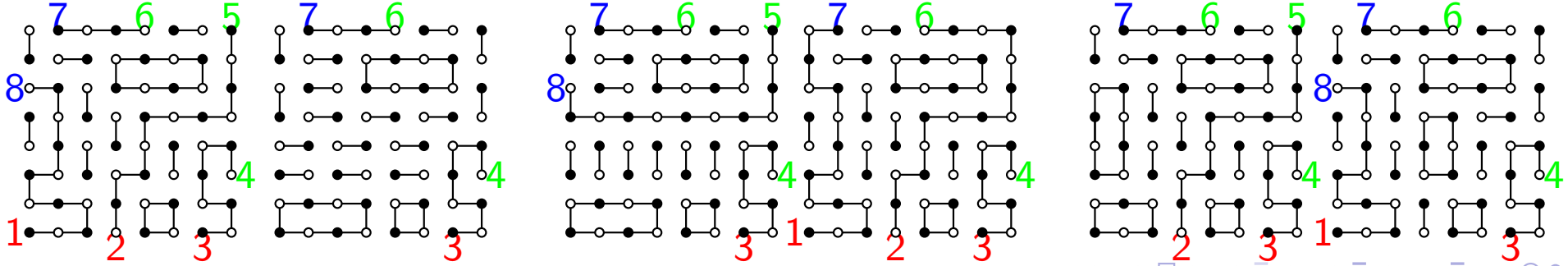
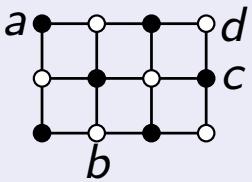


Double-dimer condensation

Theorem (Kuo04, Theorem 5.1)



Let vertices a, b, c, and d appear in a cyclic order on a face of G. If $a, c \in V_1$ and $b, d \in V_2$, then

$$Z^D(G)Z^D(G-\{a,b,c,d\})=Z^D(G-\{a,b\})Z^D(G-\{c,d\})+Z^D(G-\{a,d\})Z^D(G-\{b,c\})$$

Theorem (J.)

Let $x, y, w, v \in \mathbf{N}$ such that $x < w \in V_1$ and $y < v \in V_2$. If $\{x, y, w, v\}$ contains at least one node of each RGB color and x, y, w, v appear in cyclic order then

$$Z^{DD}_{\sigma}(G, \mathbf{N})Z^{DD}_{\sigma_{xywv}}(G - \{x, y, w, v\}, \mathbf{N} - \{x, y, w, v\}) =$$

$$Z^{DD}_{\sigma_{xy}}(G - \{x, y\}, \mathbf{N} - \{x, y\})Z^{DD}_{\sigma_{wv}}(G - \{w, v\}, \mathbf{N} - \{w, v\}) +$$

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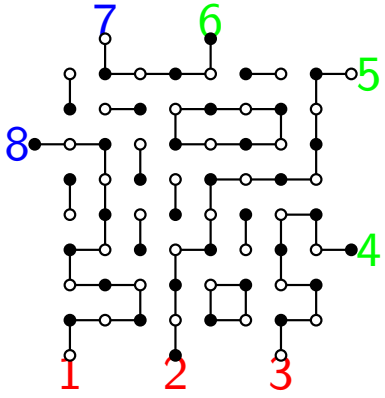

Double-dimer condensation

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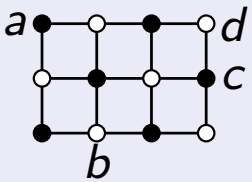

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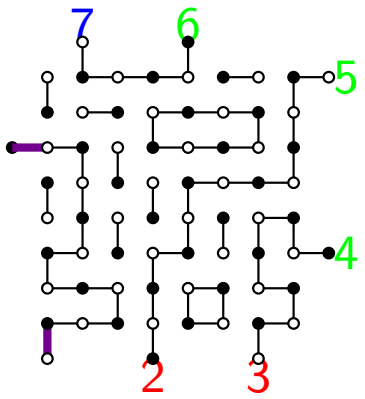


Proof of double-dimer condensation

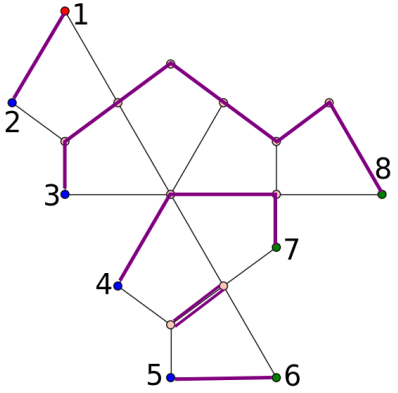
Theorem (Desnanot-Jacobi identity/Dodgson condensation)

$$\det(M)\det(M_{i,j}^{i,j}) = \det(M_i^i)\det(M_j^j) - \det(M_i^j)\det(M_j^i)$$

Theorem (J., generalization of Kenyon and Wilson, Theorem 6.1)

When σ is a tripartite pairing,

$$\frac{Z_\sigma^{DD}(G,\mathbf{N})}{(Z^D(G))^2} = sign_{OE}(\sigma)\det[1_{i,j \text{ RGB-colored differently}}\, Y_{i,j}]_{j=w_1,w_2,\ldots,w_n}^{i=b_1,b_2,\ldots,b_n}.$$

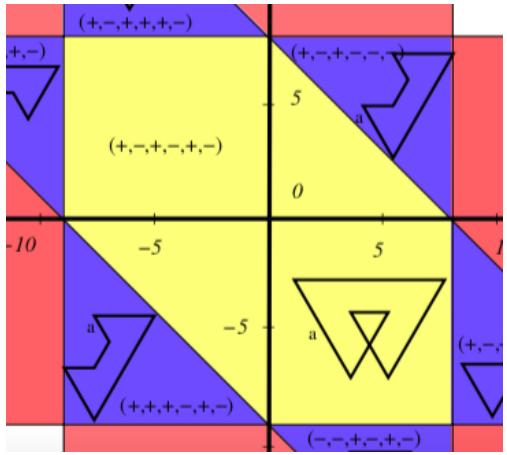


$$\frac{Z_\sigma^{DD}(G,\mathbf{N})}{(Z^D(G))^2} = \begin{vmatrix} Y_{2,1} & Y_{2,6} & Y_{2,7} & Y_{2,8} \\ Y_{3,1} & Y_{3,6} & Y_{3,7} & Y_{3,8} \\ Y_{4,1} & Y_{4,6} & Y_{4,7} & Y_{4,8} \\ Y_{5,1} & Y_{5,6} & Y_{5,7} & Y_{5,8} \end{vmatrix}$$

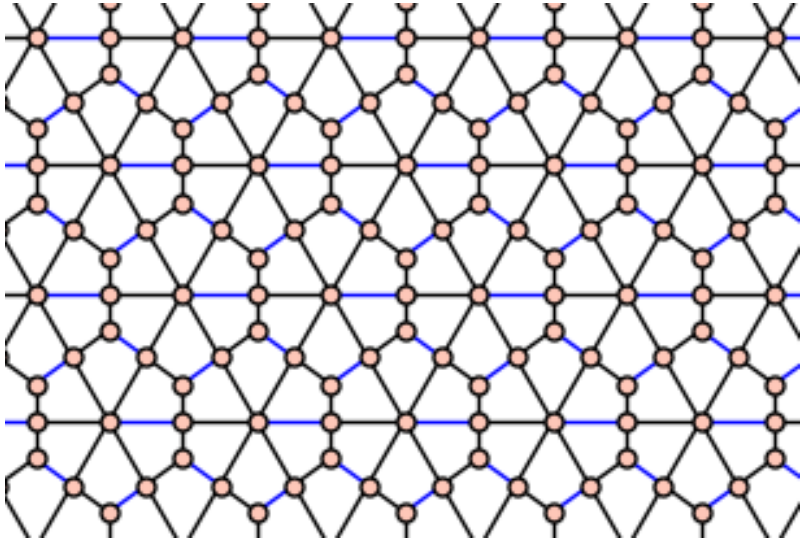
Sketch of proof for self-intersecting contours

Lemma (J-Lai-Musiker 2020+)

Let G corr. to (i, j, k) on the rim of the hexagonal region. There is a bijection between dimer configurations of G and tripartite double-dimer configurations of G with the described node set.



The bijection: Given a dimer configuration of such a graph, superimpose the following dimer configuration of the dP_3 lattice:



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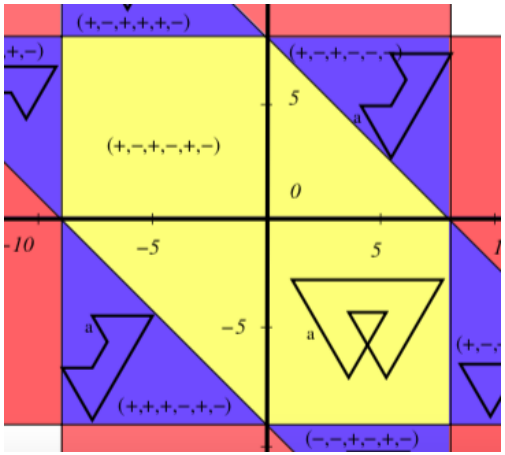

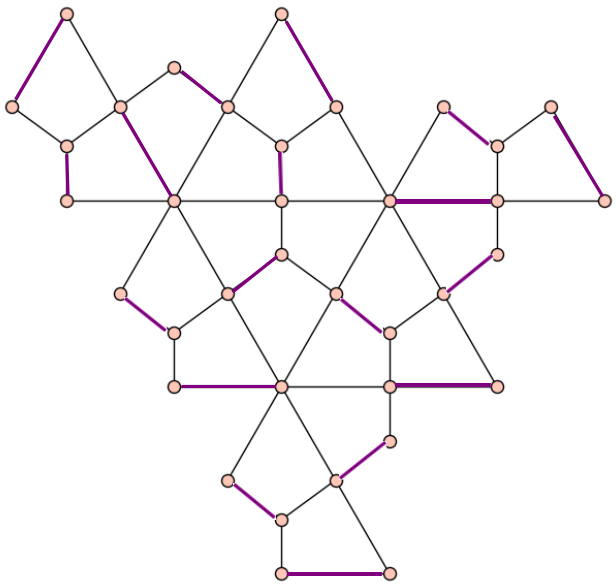

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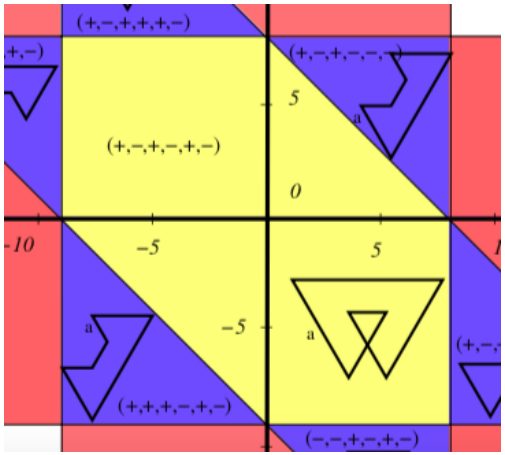

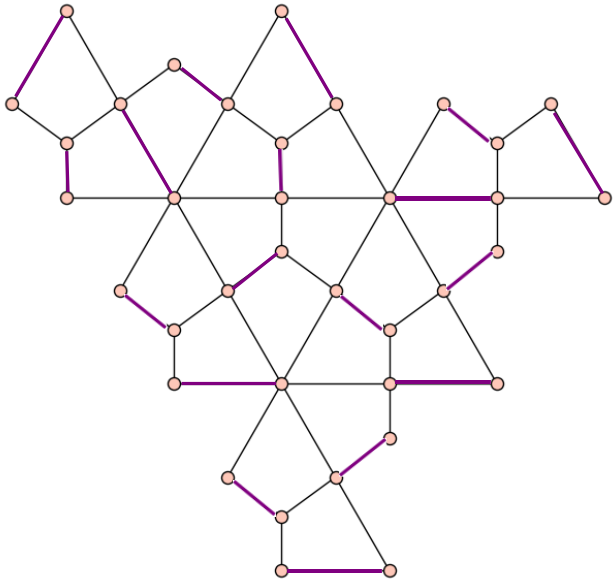

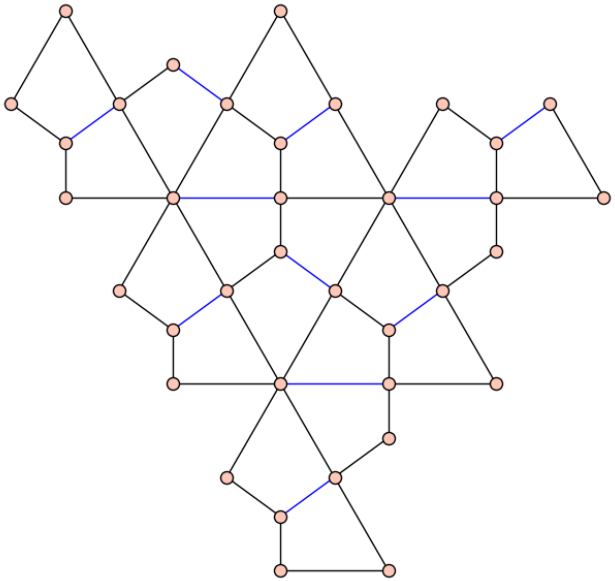

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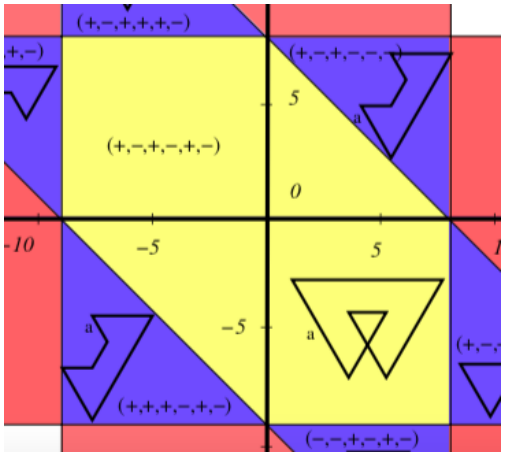
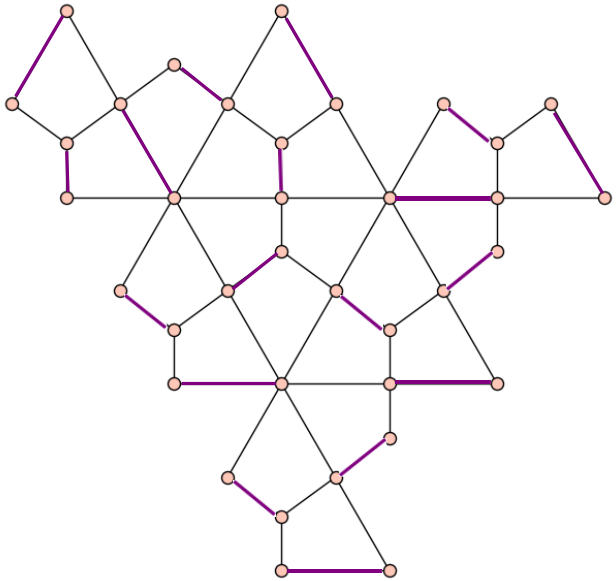
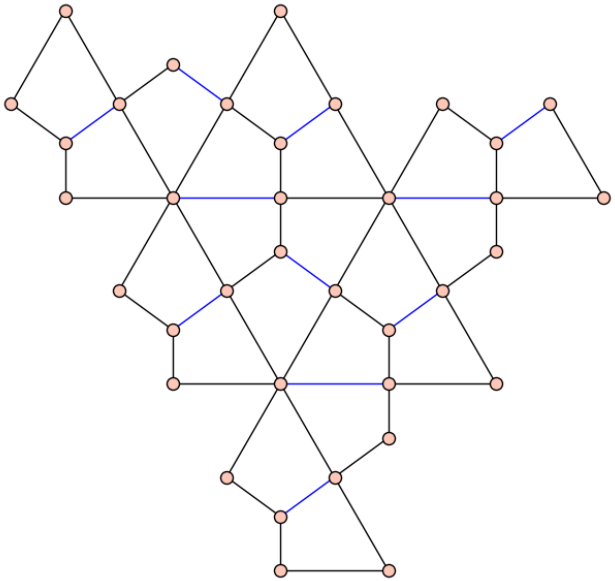
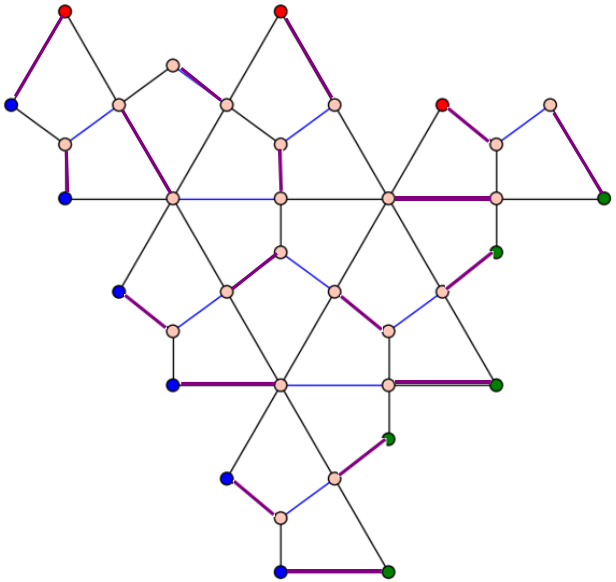






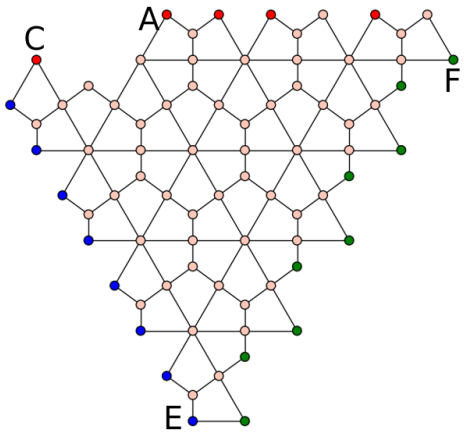


Sketch of proof for self-intersecting contours

Proof idea: Induction using double-dimer condensation.
Base case: The dimer interpretations of LM17.

$$z_{-1,-2,4} \cdot z_{0,-2,2} \quad = \quad z_{-1,-2,3} \cdot z_{0,-2,3} + z_{-1,-1,3} \cdot z_{0,-3,3}$$

$$\begin{aligned} Z^{DD}_{\sigma}(G,\mathbf{N}) Z^{DD}_{\sigma_5}(G-ACEF,\mathbf{N}-ACEF) &= Z^{DD}_{\sigma_1}(G-AC,\mathbf{N}-AC) Z^{DD}_{\sigma_2}(G-EF,\mathbf{N}-EF) \\ &\quad + Z^{DD}_{\sigma_3}(G-CE,\mathbf{N}-CE) Z^{DD}_{\sigma_4}(G-AF,\mathbf{N}-AF) \end{aligned}$$



$(-1,-2,4)$

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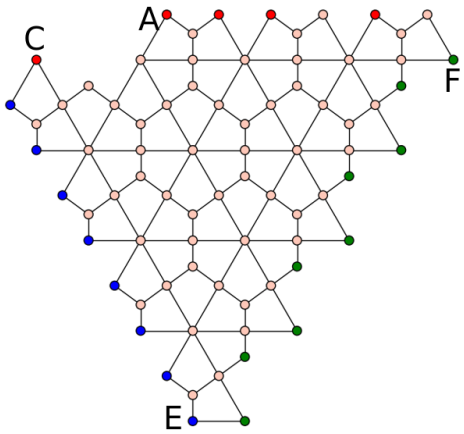

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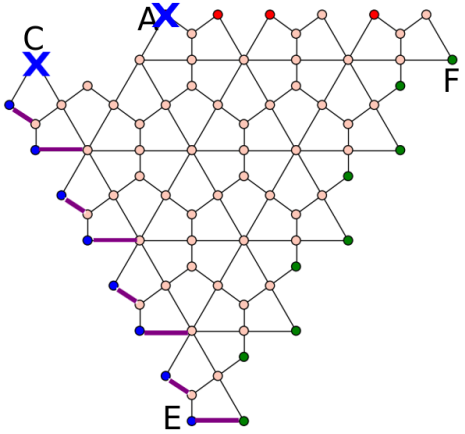

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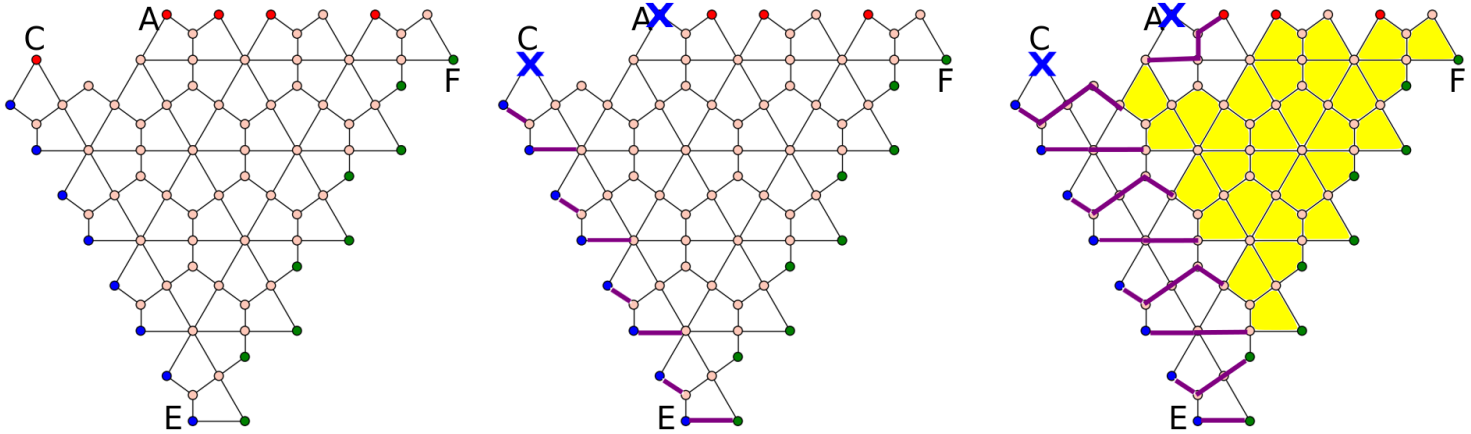

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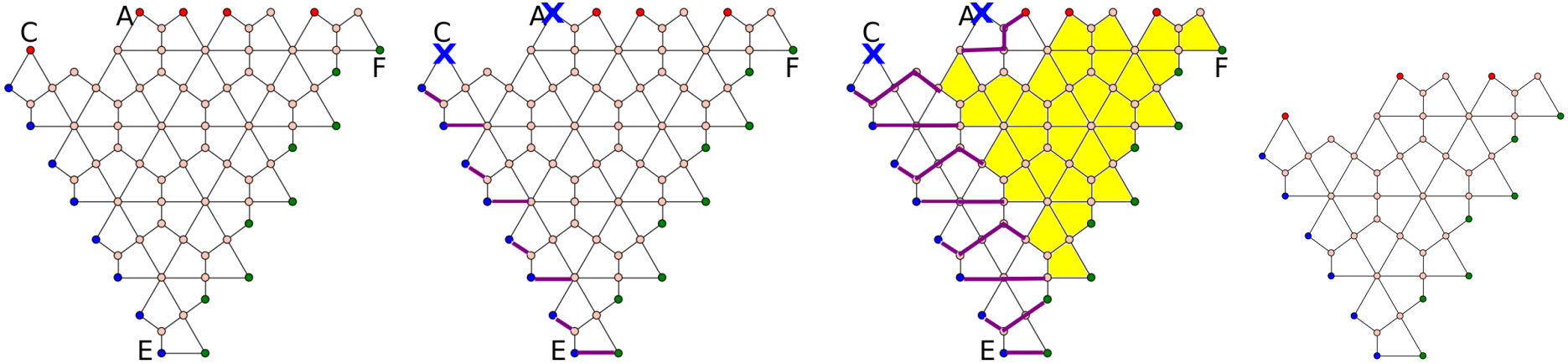

$(-1,-2,4)$ $(-1,-2,3)$

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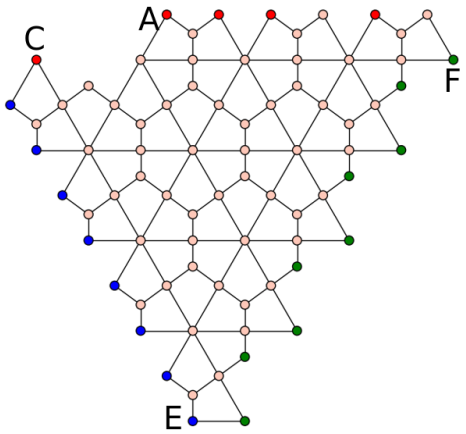
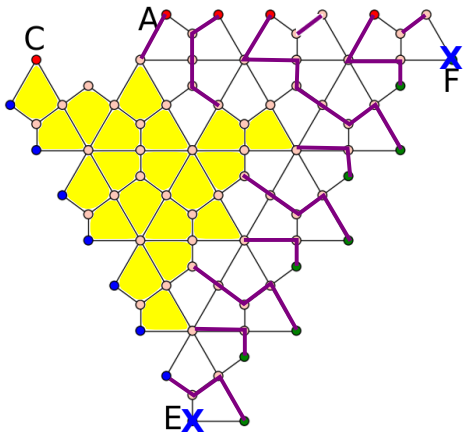
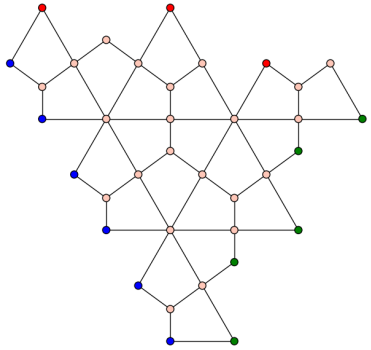


$(-1,-2,4)$





$(0,-2,3)$

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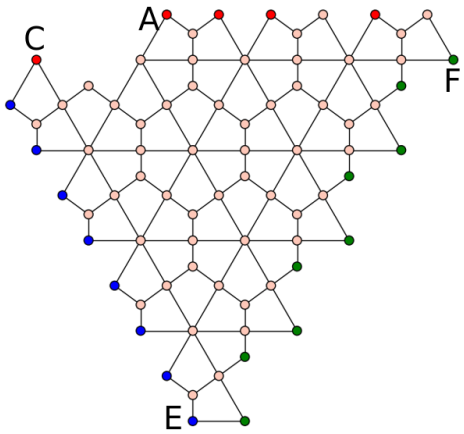
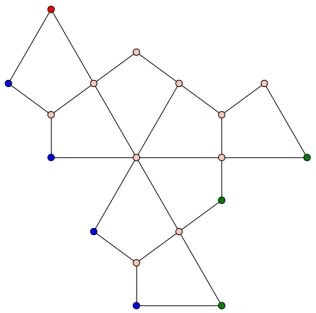
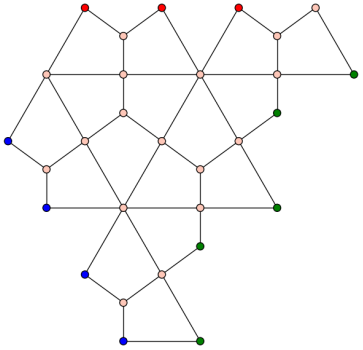


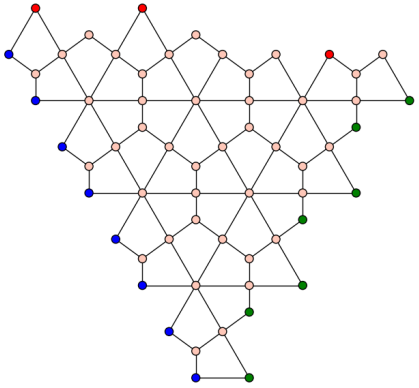
$(-1, -2, 4)$



$(0, -2, 2)$



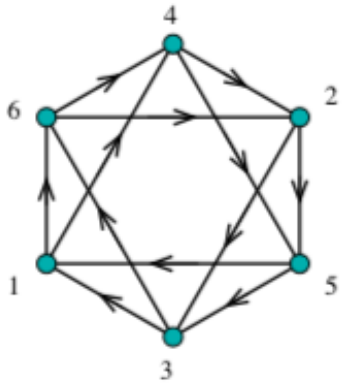
$(-1, -1, 3)$



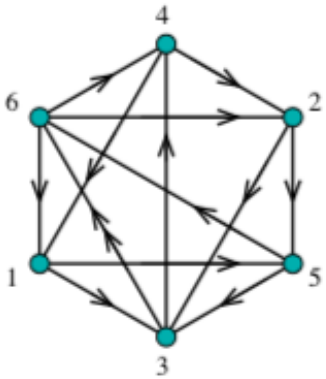
$(0, -3, 3)$

Future Work

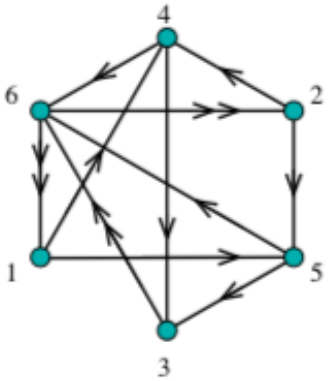
If we mutate Q by any sequence of toric mutations, we get a quiver that is graph isomorphic to one of the following:



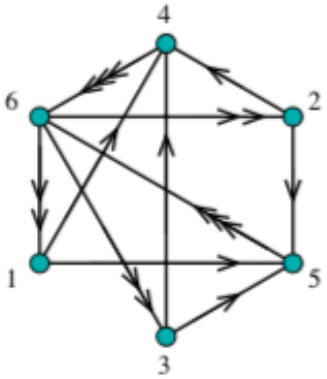
Model I



Model II



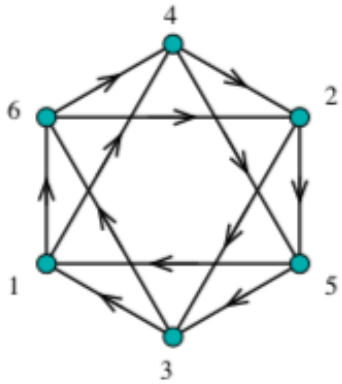
Model III



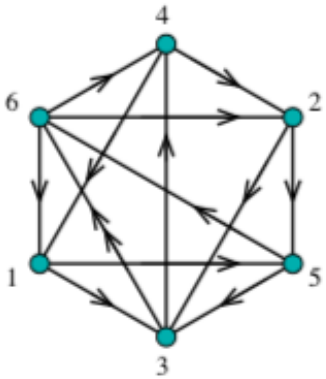
Model IV

Future Work

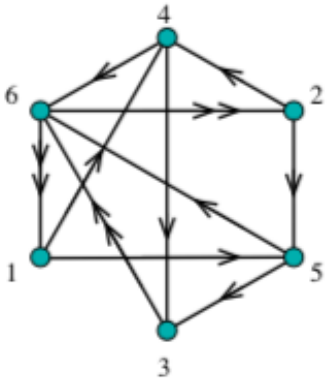
If we mutate Q by any sequence of toric mutations, we get a quiver that is graph isomorphic to one of the following:



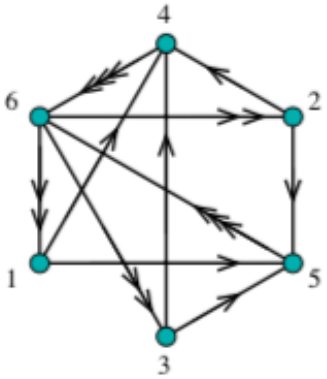
Model I



Model II



Model III



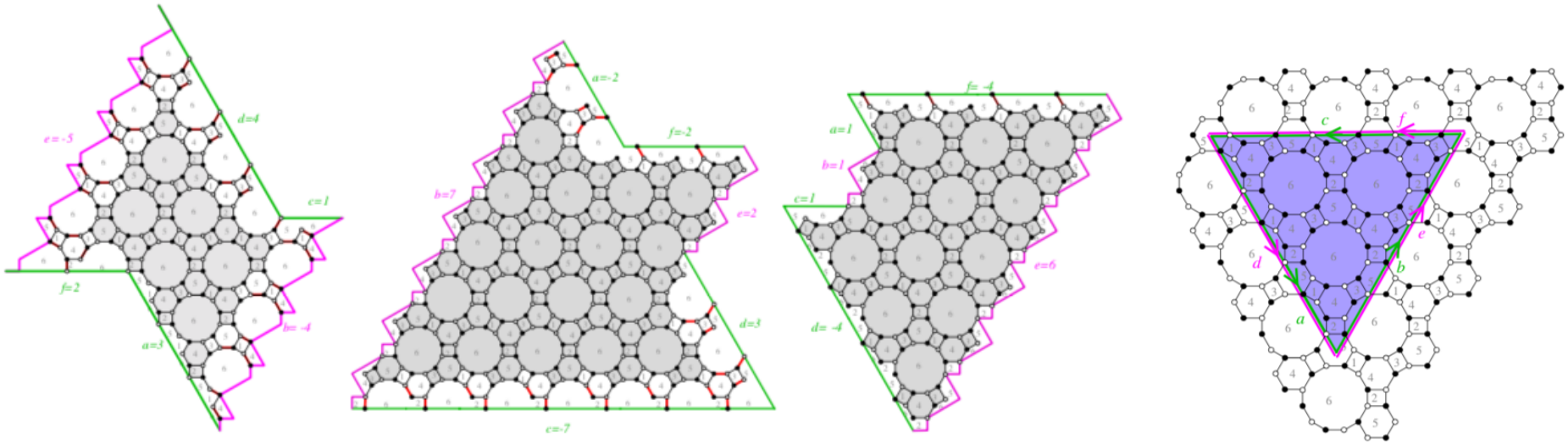
Model IV

Conjecture. For the Model II, III, and IV quivers, toric cluster variables associated to self-intersecting contours have a similar double-dimer interpretation.

Image shown is Figure 2 from T. Lai and G. Musiker, *Dungeons and Dragons: Combinatorics for the dP_3 Quiver*

Future Work

Conjecture. For the Model II, III, and IV quivers, toric cluster variables associated to self-intersecting contours have a similar double-dimer



Contours and subgraphs for Model IV

Images shown are Figure 19 and Figure 43 from T. Lai and G. Musiker, *Dungeons and Dragons: Combinatorics for the dP_3 Quiver*

Thank you for listening!

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