Combinatorics of the Double-Dimer Model

Helen Jenne

University of Oregon

FPSAC 2020 Online

July 6, 2020

This talk is being recorded

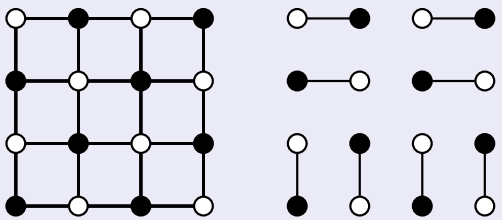
Slides available at:
`https://pages.uoregon.edu/hjenne/FPSACTalk.pdf`

Outline

The dimer model

- Today $G = (V_1, V_2, E)$ is a finite bipartite planar graph.

Definition (Dimer configuration/Perfect matching)

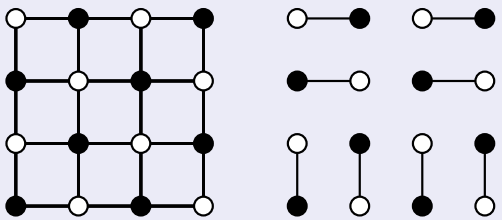


A collection of edges that covers each vertex exactly once

The dimer model

- Today $G = (V_1, V_2, E)$ is a finite bipartite planar graph.

Definition (Dimer configuration/Perfect matching)



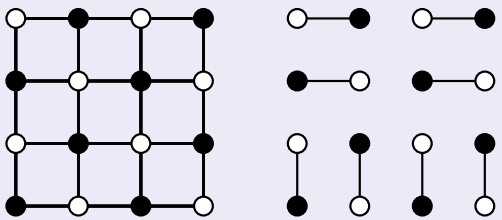
A collection of edges that covers each vertex exactly once

- Given a graph, we can assign a weight $w(e)$ to each edge.
- If M is a perfect matching (dimer configuration), $w(M) = \prod_{e \in M} w(e)$

The dimer model

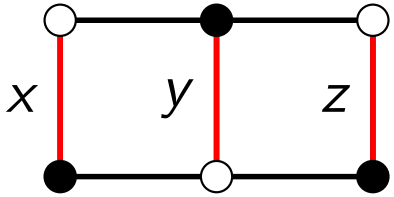
- Today $G = (V_1, V_2, E)$ is a finite bipartite planar graph.

Definition (Dimer configuration/Perfect matching)



A collection of edges that covers each vertex exactly once

- Given a graph, we can assign a weight $w(e)$ to each edge.
- If M is a perfect matching (dimer configuration), $w(M) = \prod_{e \in M} w(e)$

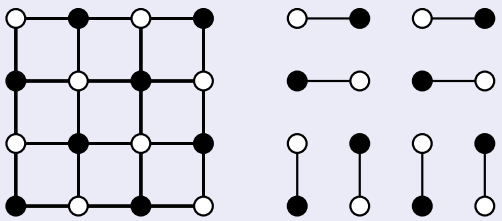


$$w(M) = xyz$$

The dimer model

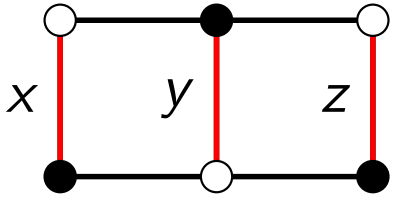
- Today $G = (V_1, V_2, E)$ is a finite bipartite planar graph.

Definition (Dimer configuration/Perfect matching)



A collection of edges that covers each vertex exactly once

- Given a graph, we can assign a weight $w(e)$ to each edge.
- If M is a perfect matching (dimer configuration), $w(M) = \prod_{e \in M} w(e)$



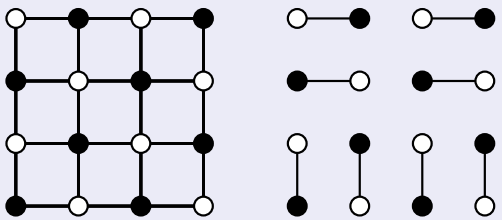
$$w(M) = xyz$$

- Let $Z^D(G) = \sum_M w(M)$, called the *partition function*.

The dimer model

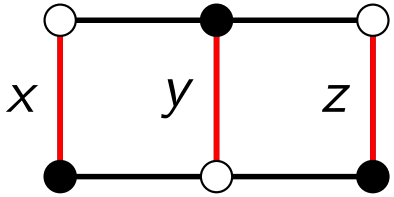
- Today $G = (V_1, V_2, E)$ is a finite bipartite planar graph.

Definition (Dimer configuration/Perfect matching)



A collection of edges that covers each vertex exactly once

- Given a graph, we can assign a weight $w(e)$ to each edge.
- If M is a perfect matching (dimer configuration), $w(M) = \prod_{e \in M} w(e)$



$$w(M) = xyz$$

- Let $Z^D(G) = \sum_M w(M)$, called the *partition function*.

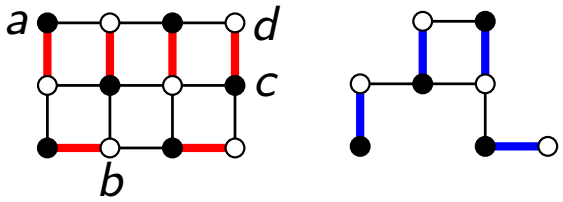
$$Z^D(G) = xyz + x + z$$

Kuo condensation

Theorem (Kuo04, Theorem 5.1)

Let vertices a, b, c, and d appear in a cyclic order on a face of G. If $a, c \in V_1$ and $b, d \in V_2$, then

$$Z^D(G)Z^D(G-\{a,b,c,d\})=Z^D(G-\{a,b\})Z^D(G-\{c,d\})+Z^D(G-\{a,d\})Z^D(G-\{b,c\})$$

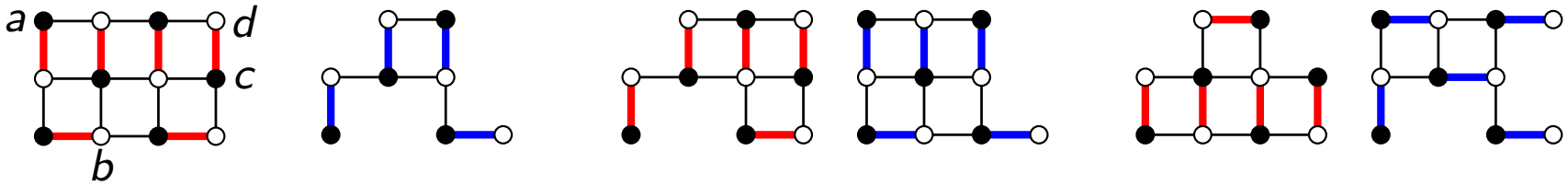


Kuo condensation

Theorem (Kuo04, Theorem 5.1)

Let vertices a, b, c, and d appear in a cyclic order on a face of G. If $a, c \in V_1$ and $b, d \in V_2$, then

$$Z^D(G)Z^D(G-\{a,b,c,d\})=Z^D(G-\{a,b\})Z^D(G-\{c,d\})+Z^D(G-\{a,d\})Z^D(G-\{b,c\})$$

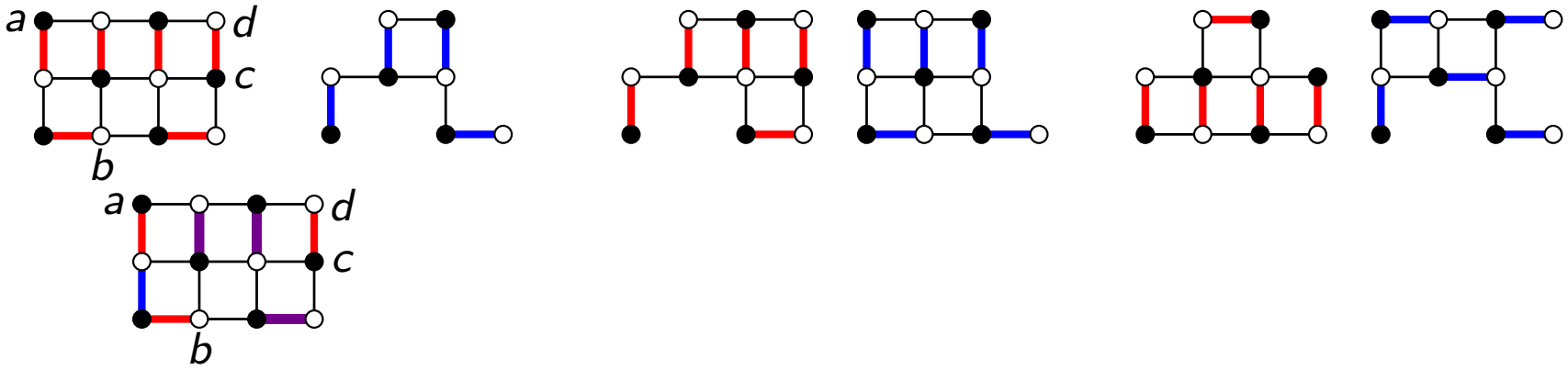


Kuo condensation

Theorem (Kuo04, Theorem 5.1)

Let vertices a, b, c, and d appear in a cyclic order on a face of G. If $a, c \in V_1$ and $b, d \in V_2$, then

$$Z^D(G)Z^D(G-\{a,b,c,d\})=Z^D(G-\{a,b\})Z^D(G-\{c,d\})+Z^D(G-\{a,d\})Z^D(G-\{b,c\})$$

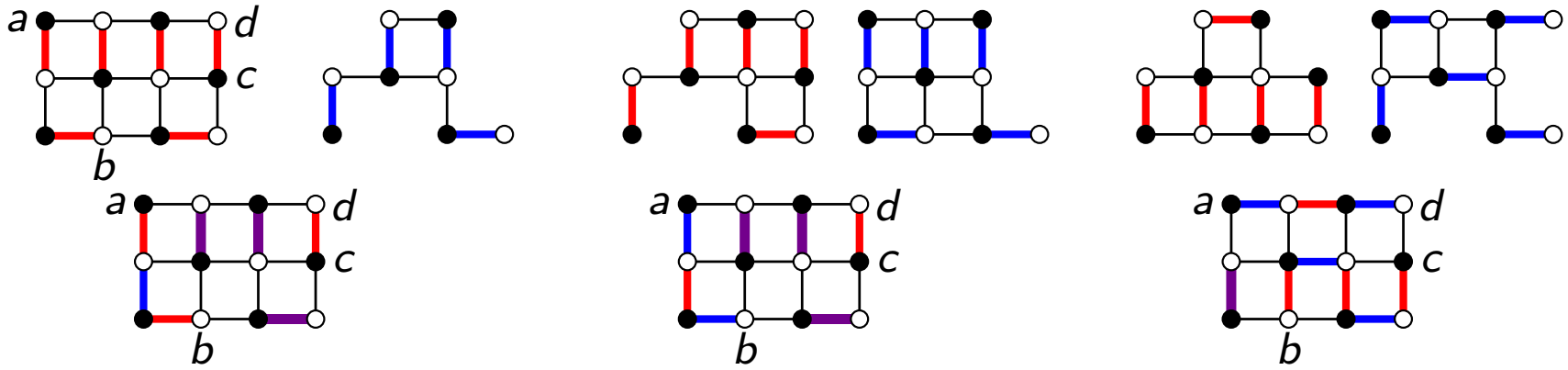


Kuo condensation

Theorem (Kuo04, Theorem 5.1)

Let vertices a, b, c, and d appear in a cyclic order on a face of G. If $a, c \in V_1$ and $b, d \in V_2$, then

$$Z^D(G)Z^D(G-\{a,b,c,d\})=Z^D(G-\{a,b\})Z^D(G-\{c,d\})+Z^D(G-\{a,d\})Z^D(G-\{b,c\})$$

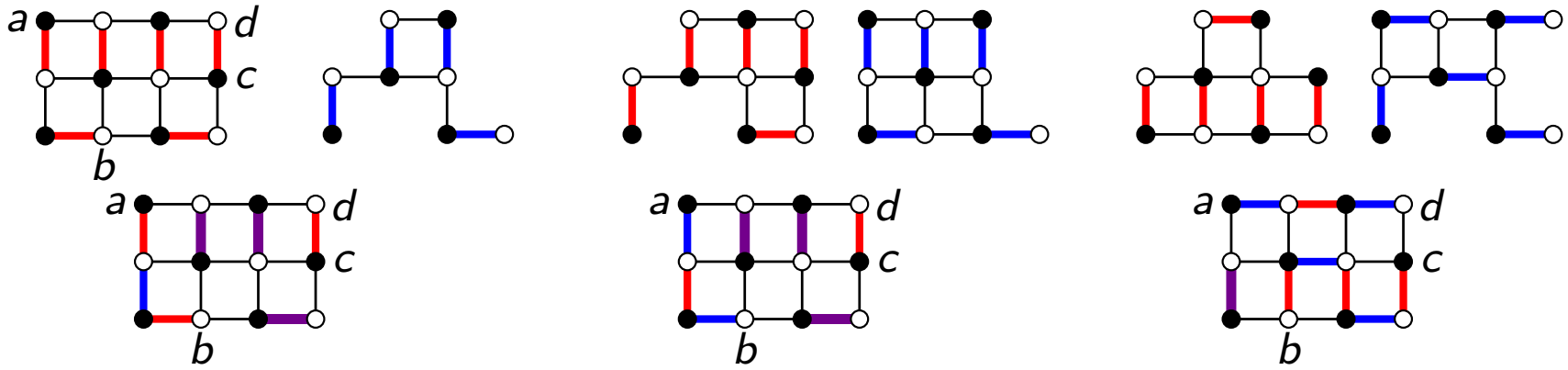


Kuo condensation

Theorem (Kuo04, Theorem 5.1)

Let vertices a, b, c, and d appear in a cyclic order on a face of G. If $a, c \in V_1$ and $b, d \in V_2$, then

$$Z^D(G)Z^D(G-\{a,b,c,d\})=Z^D(G-\{a,b\})Z^D(G-\{c,d\})+Z^D(G-\{a,d\})Z^D(G-\{b,c\})$$



Examples of non-bijective proofs:

– Fulmek, *Graphical condensation, overlapping Pfaffians and superpositions of Matchings*

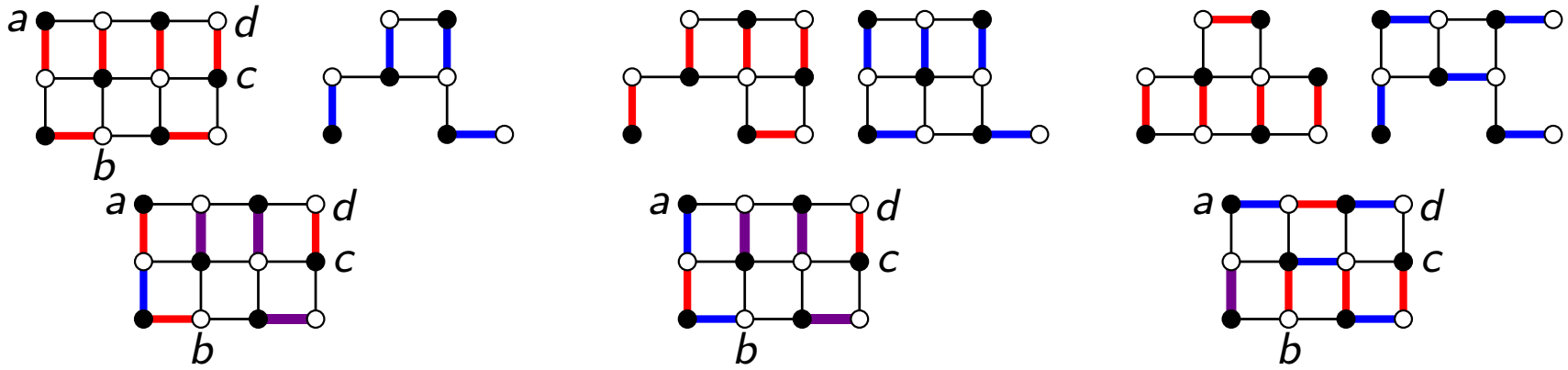
– Speyer, *Variations on a theme of Kasteleyn, with Application to the TNN Grassmannian*

Kuo condensation

Theorem (Kuo04, Theorem 5.1)

Let vertices a, b, c, and d appear in a cyclic order on a face of G. If $a, c \in V_1$ and $b, d \in V_2$, then

$$Z^D(G)Z^D(G-\{a,b,c,d\})=Z^D(G-\{a,b\})Z^D(G-\{c,d\})+Z^D(G-\{a,d\})Z^D(G-\{b,c\})$$



Examples of non-bijective proofs:

– Fulmek, *Graphical condensation, overlapping Pfaffians and superpositions of Matchings*

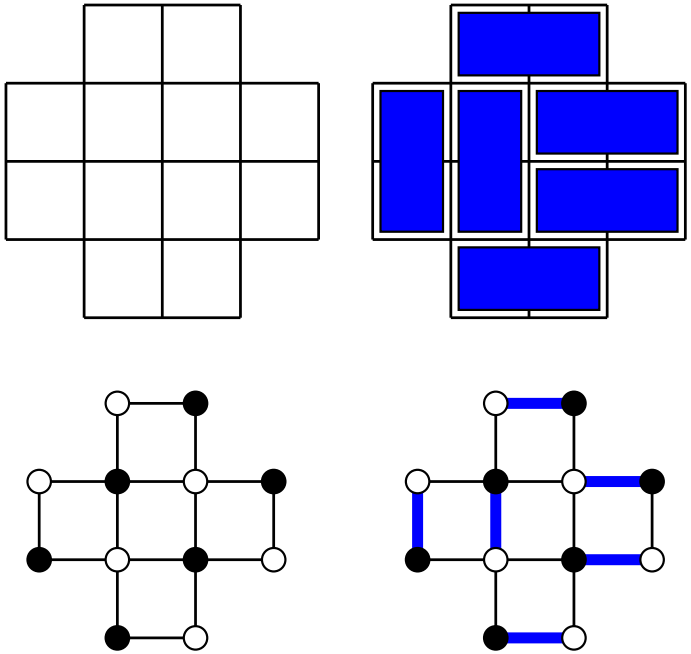
– Speyer, *Variations on a theme of Kasteleyn, with Application to the TNN Grassmannian*

Theorem (Desnanot-Jacobi identity/Dodgson condensation)

$$\det(M)\det(M_{i,j}^{i,j}) = \det(M_i^i)\det(M_j^j) - \det(M_i^j)\det(M_j^i)$$

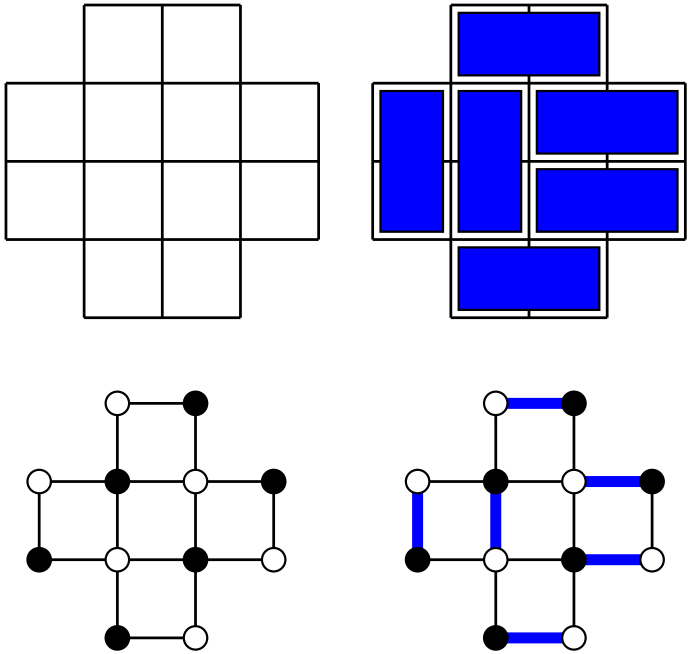
Applications of Kuo's work

- Tiling enumeration
  New proof that the number of tilings of the order-n Aztec diamond is $2^{n(n+1)/2}$ (EKLP92)



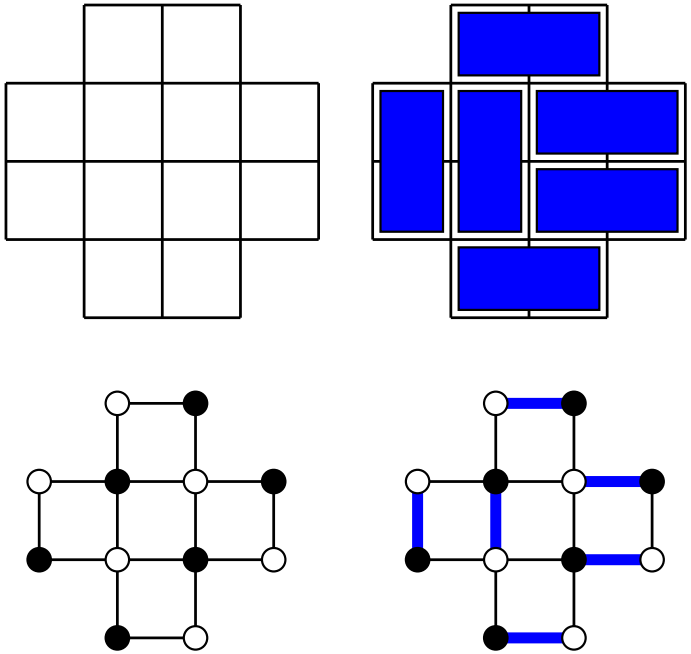
Applications of Kuo's work

- Tiling enumeration
  New proof that the number of tilings of the order-n Aztec diamond is $2^{n(n+1)/2}$ (EKLP92)
- Cluster algebras
  (LM17) Toric cluster variables for the quiver associated to the cone of the del Pezzo surface of degree 6



Applications of Kuo's work

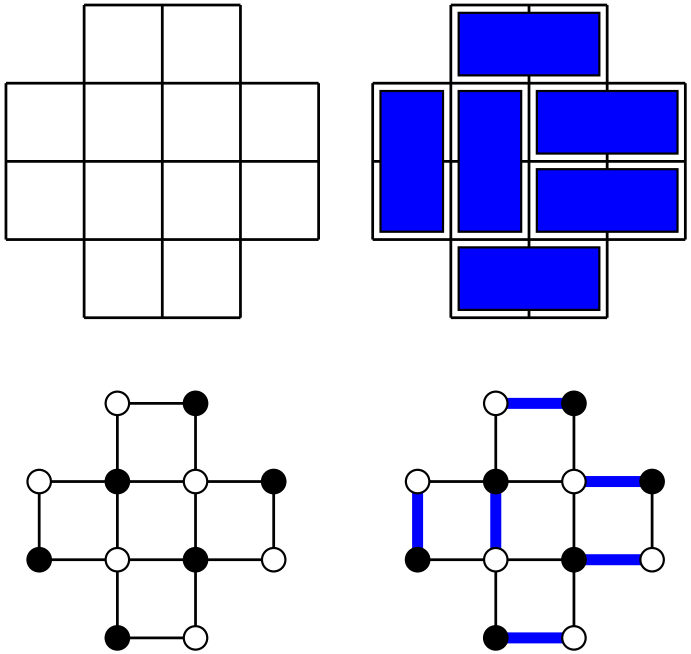
- Tiling enumeration
  New proof that the number of tilings of the order-n Aztec diamond is $2^{n(n+1)/2}$ (EKLP92)
- Cluster algebras
  (LM17) Toric cluster variables for the quiver associated to the cone of the del Pezzo surface of degree 6



Main result. An analogue of Kuo's theorem for double-dimer configs.

Applications of Kuo's work

- Tiling enumeration
  New proof that the number of tilings of the order-n Aztec diamond is $2^{n(n+1)/2}$ (EKLP92)
- Cluster algebras
  (LM17) Toric cluster variables for the quiver associated to the cone of the del Pezzo surface of degree 6



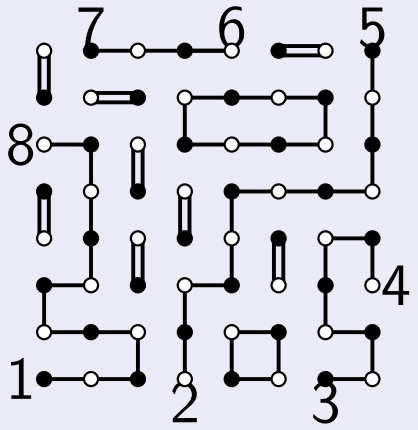
Main result. An analogue of Kuo's theorem for double-dimer configs.

Application: A problem in Donaldson-Thomas theory and Pandharipande-Thomas theory (joint work with Ben Young and Gautam Webb).

Double-dimer configurations

N is a set of special vertices called *nodes* on the outer face of G.

Definition (Double-dimer configuration on $(G, \mathbf{N})$)



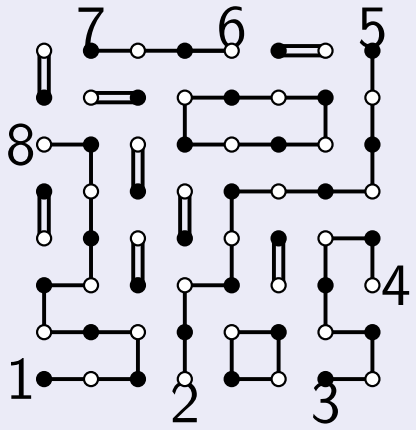
Configuration of

- ℓ disjoint loops
- Doubled edges
- Paths connecting nodes in pairs

Double-dimer configurations

N is a set of special vertices called *nodes* on the outer face of G.

Definition (Double-dimer configuration on $(G, \mathbf{N})$)



Configuration of

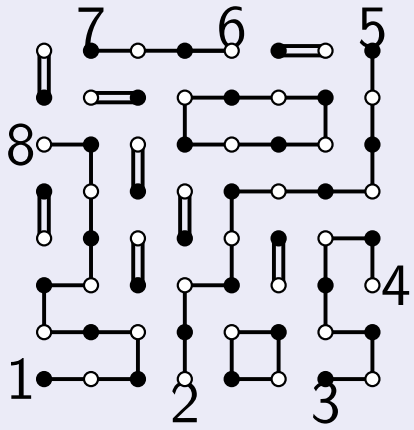
- ℓ disjoint loops
- Doubled edges
- Paths connecting nodes in pairs

Its weight is the product of its edge weights $\times\ 2^{\ell}$

Double-dimer configurations

N is a set of special vertices called *nodes* on the outer face of G.

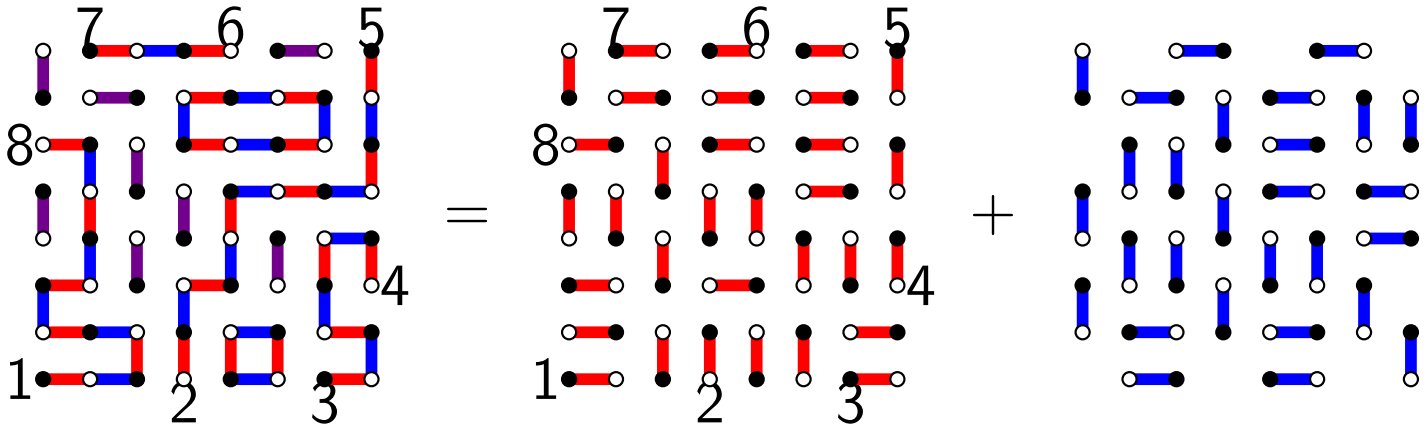
Definition (Double-dimer configuration on $(G, \mathbf{N})$)



Configuration of

- ℓ disjoint loops
- Doubled edges
- Paths connecting nodes in pairs

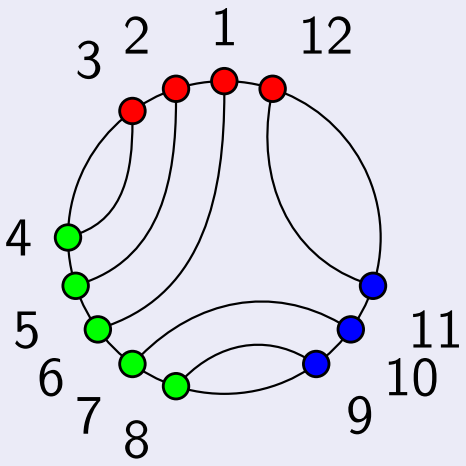
Its weight is the product of its edge weights $\times\ 2^{\ell}$



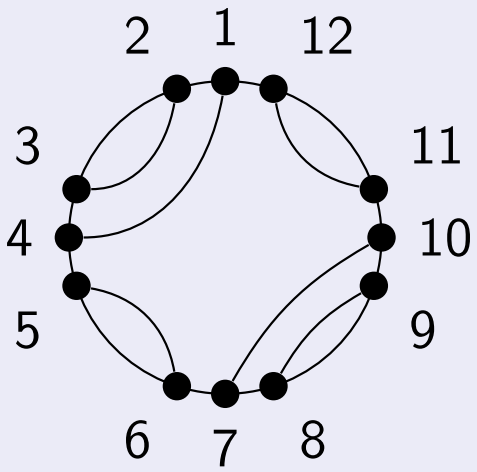
Tripartite pairings

Definition (Tripartite pairing)

A planar pairing σ of $\mathbf{N}$ is *tripartite* if the nodes can be divided into ≤ 3 sets of circularly consecutive nodes so that no node is paired with a node in the same set.



Tripartite



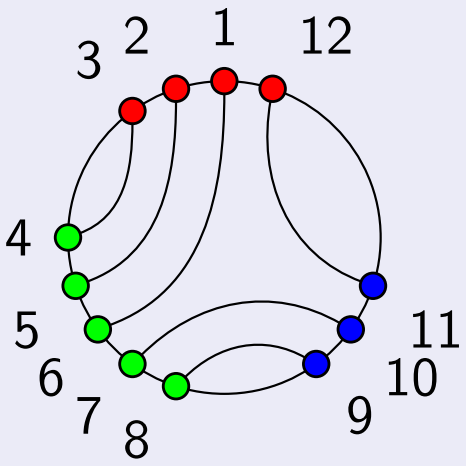
Not tripartite

We often color the nodes in the sets red, green, and blue, in which case σ has no monochromatic pairs.

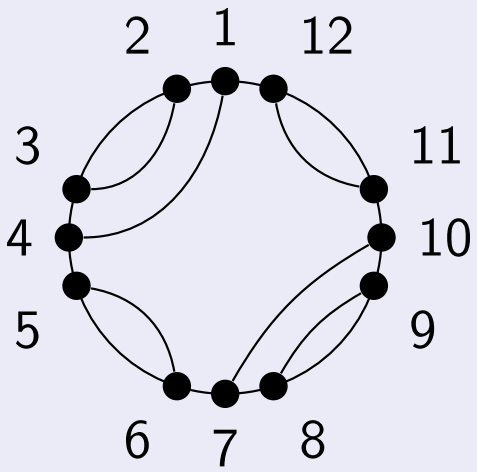
Tripartite pairings

Definition (Tripartite pairing)

A planar pairing σ of $\mathbf{N}$ is *tripartite* if the nodes can be divided into ≤ 3 sets of circularly consecutive nodes so that no node is paired with a node in the same set.



Tripartite



Not tripartite

We often color the nodes in the sets red, green, and blue, in which case σ has no monochromatic pairs.

Dividing nodes into three sets R, G, and B defines a tripartite pairing.

Main Result

$Z_\sigma^{DD}(G, \mathbf{N})$ denotes the weighted sum of all DD config with pairing σ.

Main Result

$Z_\sigma^{DD}(G, \mathbf{N})$ denotes the weighted sum of all DD config with pairing σ.

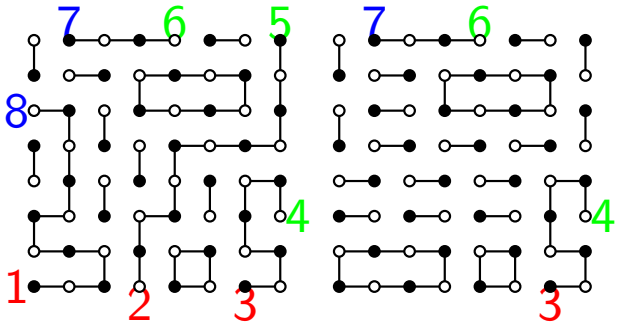
Theorem (J.)

Divide $\mathbf{N}$ *into sets* R, G, *and* B *and let* σ *be the corresponding tripartite pairing. Let* x, y, w, v *be nodes appearing in a cyclic order such that* $\{x, y, w, v\}$ *contains at least one node of each RGB color. If* $x, w \in V_1$ *and* $y, v \in V_2$ *then*

$$Z_\sigma^{DD}(\mathbf{N}) Z_{\sigma_5}^{DD}(\mathbf{N} - \{x, y, w, v\}) = Z_{\sigma_1}^{DD}(\mathbf{N} - \{x, y\}) Z_{\sigma_2}^{DD}(\mathbf{N} - \{w, v\}) + Z_{\sigma_3}^{DD}(\mathbf{N} - \{x, v\}) Z_{\sigma_4}^{DD}(\mathbf{N} - \{w, y\})$$

Example.

$$Z_\sigma^{DD}(\mathbf{N}) Z_{\sigma_5}^{DD}(\mathbf{N} - 1, 2, 5, 8) = Z_{\sigma_1}^{DD}(\mathbf{N} - 1, 8) Z_{\sigma_2}^{DD}(\mathbf{N} - 2, 5) + Z_{\sigma_3}^{DD}(\mathbf{N} - 1, 2) Z_{\sigma_4}^{DD}(\mathbf{N} - 5, 8)$$



Main Result

$Z_\sigma^{DD}(G, \mathbf{N})$ denotes the weighted sum of all DD config with pairing σ.

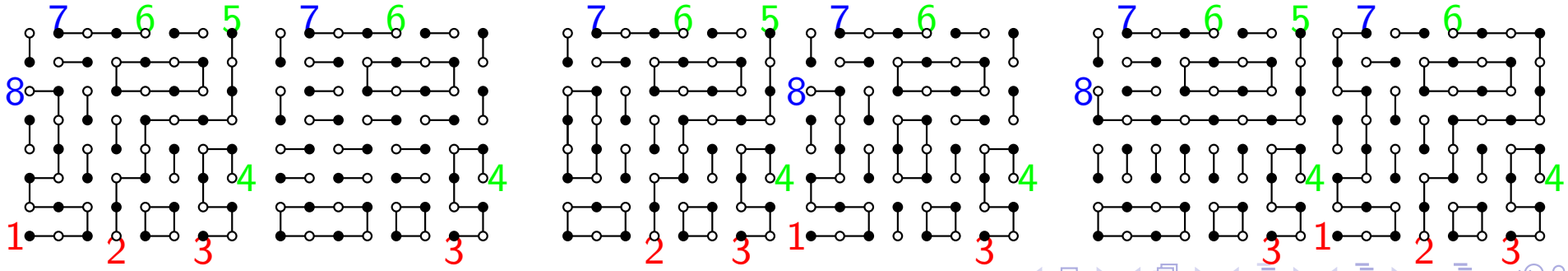
Theorem (J.)

Divide $\mathbf{N}$ *into sets* R, G, *and* B *and let* σ *be the corresponding tripartite pairing. Let* x, y, w, v *be nodes appearing in a cyclic order such that* $\{x, y, w, v\}$ *contains at least one node of each RGB color. If* $x, w \in V_1$ *and* $y, v \in V_2$ *then*

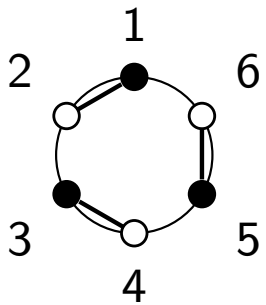
$$Z_\sigma^{DD}(\mathbf{N}) Z_{\sigma_5}^{DD}(\mathbf{N} - \{x, y, w, v\}) = Z_{\sigma_1}^{DD}(\mathbf{N} - \{x, y\}) Z_{\sigma_2}^{DD}(\mathbf{N} - \{w, v\}) + Z_{\sigma_3}^{DD}(\mathbf{N} - \{x, v\}) Z_{\sigma_4}^{DD}(\mathbf{N} - \{w, y\})$$

Example.

$$Z_\sigma^{DD}(\mathbf{N}) Z_{\sigma_5}^{DD}(\mathbf{N} - 1, 2, 5, 8) = Z_{\sigma_1}^{DD}(\mathbf{N} - 1, 8) Z_{\sigma_2}^{DD}(\mathbf{N} - 2, 5) + Z_{\sigma_3}^{DD}(\mathbf{N} - 1, 2) Z_{\sigma_4}^{DD}(\mathbf{N} - 5, 8)$$



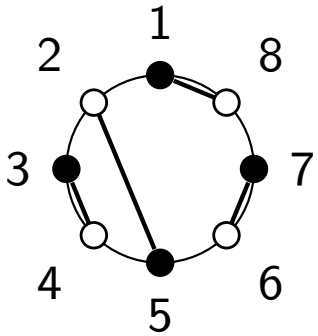
Background: Double-dimer pairing probabilities

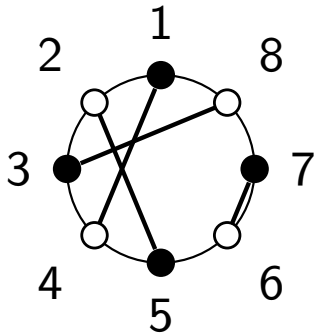


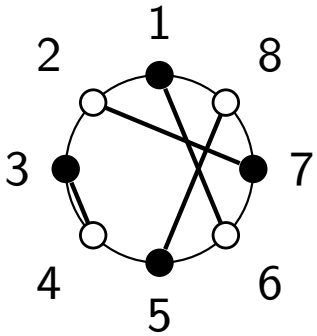
$$\widehat{\Pr}\left(\begin{matrix}1\\2\end{matrix}\middle|\begin{matrix}3\\4\end{matrix}\middle|\begin{matrix}5\\6\end{matrix}\right) = X_{1,4}X_{2,5}X_{3,6} + X_{1,2}X_{3,4}X_{5,6}$$

$X_{i,j}$ is a ratio of dimer partition functions.

Precisely, $X_{i,j} = \frac{Z^D(G^{BW}_{i,j})}{Z^D(G^{BW})}$, where $G^{BW} \subseteq G$ contains a certain subset of $\mathbf{N}$.







$$\widehat{\Pr}\left(\begin{matrix}1\\8\end{matrix}\middle|\begin{matrix}3\\4\end{matrix}\middle|\begin{matrix}5\\2\end{matrix}\middle|\begin{matrix}7\\6\end{matrix}\right) = X_{1,8}X_{3,4}X_{5,2}X_{7,6} - X_{1,4}X_{3,8}X_{5,2}X_{7,6} + X_{1,6}X_{3,4}X_{5,8}X_{7,2}$$
$$-X_{1,8}X_{3,6}X_{5,2}X_{7,4} - X_{1,4}X_{3,6}X_{5,8}X_{7,2} + X_{1,6}X_{3,8}X_{5,2}X_{7,4}$$

Theorem (KW11a, Theorem 1.3)

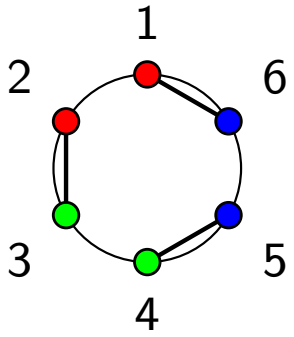
$\widehat{Pr}(\sigma)$ is an integer-coeff homogeneous polynomial in the quantities $X_{i,j}$

Background: Determinant formula

Theorem (KW09, Theorem 6.1)

When σ is a tripartite pairing,

$$\widehat{Pr}(\sigma) = \det[1_{i,j \text{ RGB-colored differently}} X_{i,j}]_{j=\sigma(1),\sigma(3),\ldots,\sigma(2n-1)}^{i=1,3,\ldots,2n-1}.$$



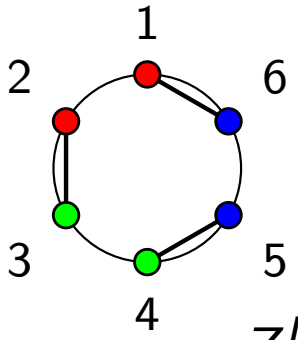
$$\widehat{\mathrm{Pr}}\left(\begin{array}{c|c|c} 1 & 3 & 5 \\ 6 & 2 & 4 \end{array}\right) = \begin{vmatrix} X_{1,6} & 0 & X_{1,4} \\ X_{3,6} & X_{3,2} & 0 \\ 0 & X_{5,2} & X_{5,4} \end{vmatrix}$$

Background: Determinant formula

Theorem (KW09, Theorem 6.1)

When σ is a tripartite pairing,

$$\widehat{Pr}(\sigma) = \det[1_{i,j \text{ RGB-colored differently}} X_{i,j}]_{j=\sigma(1),\sigma(3),\ldots,\sigma(2n-1)}^{i=1,3,\ldots,2n-1}.$$



$$\widehat{\mathrm{Pr}}\left(\begin{array}{c|c|c} 1 & 3 & 5 \\ 6 & 2 & 4 \end{array}\right) = \begin{vmatrix} X_{1,6} & 0 & X_{1,4} \\ X_{3,6} & X_{3,2} & 0 \\ 0 & X_{5,2} & X_{5,4} \end{vmatrix}$$

Since $\widehat{\mathrm{Pr}}(\sigma) := \dfrac{Z_\sigma^{DD}(G, \mathbf{N})}{(Z^D(G^{BW}))^2}$, the idea of the proof is to combine K-W's matrix with the Desnanot-Jacobi identity:

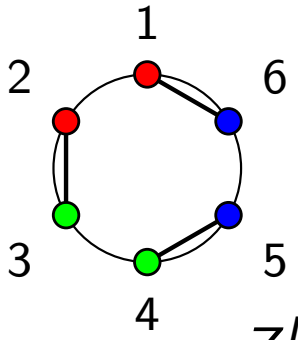
$$\det(M)\det(M_{i,j}^{i,j}) = \det(M_i^i)\det(M_j^j) - \det(M_i^j)\det(M_j^i)$$

Background: Determinant formula

Theorem (KW09, Theorem 6.1)

When σ is a tripartite pairing,

$$\widehat{Pr}(\sigma) = \det[1_{i,j \text{ RGB-colored differently}} X_{i,j}]_{j=\sigma(1),\sigma(3),\ldots,\sigma(2n-1)}^{i=1,3,\ldots,2n-1}.$$



$$\widehat{\mathrm{Pr}}\left(\begin{array}{c|c|c} 1 & 3 & 5 \\ 6 & 2 & 4 \end{array}\right) = \begin{vmatrix} X_{1,6} & 0 & X_{1,4} \\ X_{3,6} & X_{3,2} & 0 \\ 0 & X_{5,2} & X_{5,4} \end{vmatrix}$$

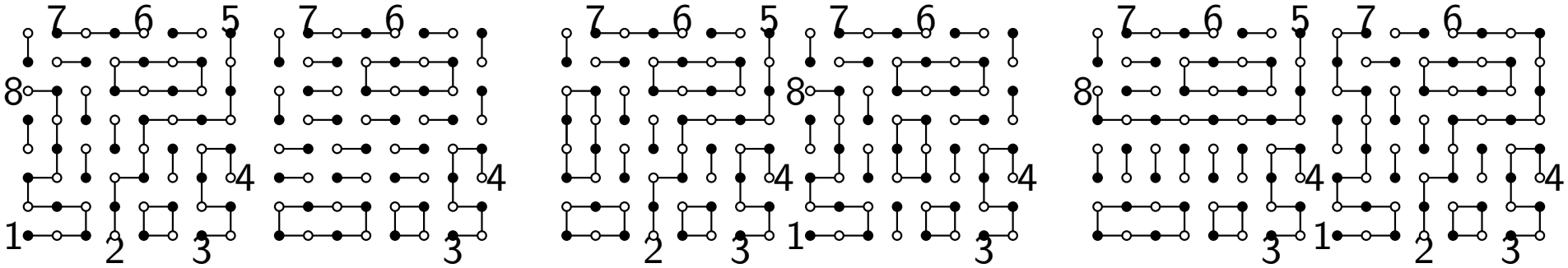
Since $\widehat{\mathrm{Pr}}(\sigma) := \dfrac{Z_\sigma^{DD}(G, \mathbf{N})}{(Z^D(G^{BW}))^2}$, the idea of the proof is to combine K-W's matrix with the Desnanot-Jacobi identity:

$$\det(M)\det(M_{i,j}^{i,j}) = \det(M_i^i)\det(M_j^j) - \det(M_i^j)\det(M_j^i)$$

The problem: Kenyon and Wilson assumed all nodes are black and odd or white and even.

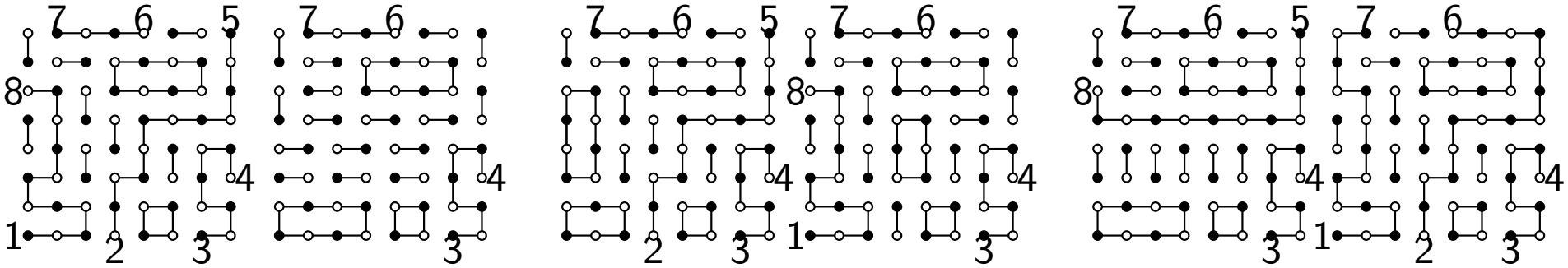
Example

$$Z_{\sigma}^{DD}(\mathbf{N})Z_{\sigma_5}^{DD}(\mathbf{N}-1,2,5,8) = Z_{\sigma_1}^{DD}(\mathbf{N}-1,8)Z_{\sigma_2}^{DD}(\mathbf{N}-2,5) + Z_{\sigma_3}^{DD}(\mathbf{N}-1,2)Z_{\sigma_4}^{DD}(\mathbf{N}-5,8)$$



Example

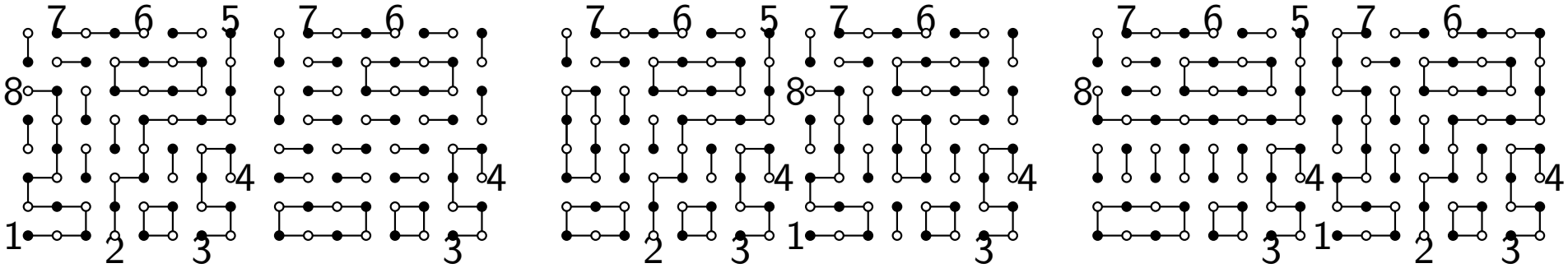
$$Z_{\sigma}^{DD}(\mathbf{N})Z_{\sigma_5}^{DD}(\mathbf{N}-1,2,5,8) = Z_{\sigma_1}^{DD}(\mathbf{N}-1,8)Z_{\sigma_2}^{DD}(\mathbf{N}-2,5) + Z_{\sigma_3}^{DD}(\mathbf{N}-1,2)Z_{\sigma_4}^{DD}(\mathbf{N}-5,8)$$



$$M = \begin{pmatrix} X_{1,8} & X_{1,4} & 0 & X_{1,6} \\ X_{3,8} & X_{3,4} & 0 & X_{3,6} \\ X_{5,8} & 0 & X_{5,2} & 0 \\ 0 & X_{7,4} & X_{7,2} & X_{7,6} \end{pmatrix}$$

Example

$$Z_{\sigma}^{DD}(\mathbf{N})Z_{\sigma_5}^{DD}(\mathbf{N}-1,2,5,8) = Z_{\sigma_1}^{DD}(\mathbf{N}-1,8)Z_{\sigma_2}^{DD}(\mathbf{N}-2,5) + Z_{\sigma_3}^{DD}(\mathbf{N}-1,2)Z_{\sigma_4}^{DD}(\mathbf{N}-5,8)$$

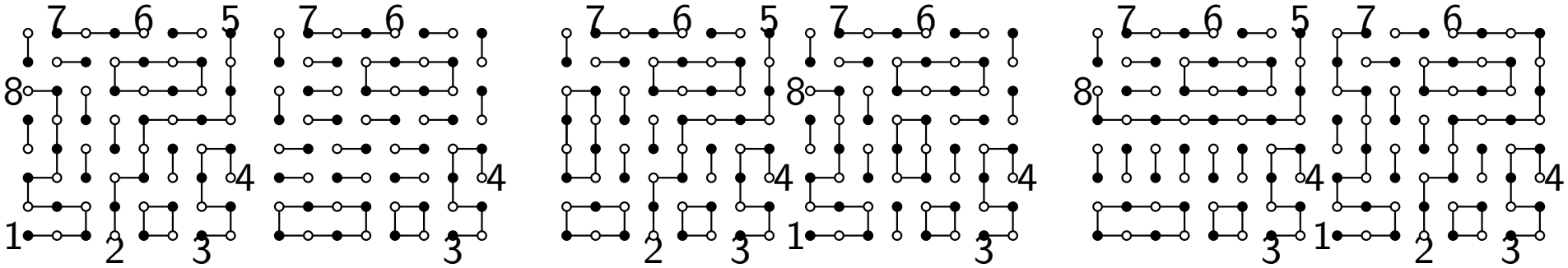


$$M = \begin{pmatrix} X_{1,8} & X_{1,4} & 0 & X_{1,6} \\ X_{3,8} & X_{3,4} & 0 & X_{3,6} \\ X_{5,8} & 0 & X_{5,2} & 0 \\ 0 & X_{7,4} & X_{7,2} & X_{7,6} \end{pmatrix}$$

$$\det(M)\det(M_{1,3}^{1,3}) = \det(M_1^1)\det(M_3^3) - \det(M_1^3)\det(M_3^1)$$

Example

$$Z_{\sigma}^{DD}(\mathbf{N})Z_{\sigma_5}^{DD}(\mathbf{N}-1,2,5,8) = Z_{\sigma_1}^{DD}(\mathbf{N}-1,8)Z_{\sigma_2}^{DD}(\mathbf{N}-2,5) + Z_{\sigma_3}^{DD}(\mathbf{N}-1,2)Z_{\sigma_4}^{DD}(\mathbf{N}-5,8)$$

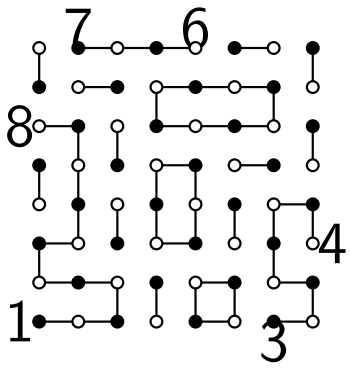


$$M = \begin{pmatrix} X_{1,8} & X_{1,4} & 0 & X_{1,6} \\ X_{3,8} & X_{3,4} & 0 & X_{3,6} \\ X_{5,8} & 0 & X_{5,2} & 0 \\ 0 & X_{7,4} & X_{7,2} & X_{7,6} \end{pmatrix}$$

$$\det(M)\det(M_{1,3}^{1,3}) = \det(M_1^1)\det(M_3^3) - \det(M_1^3)\det(M_3^1)$$

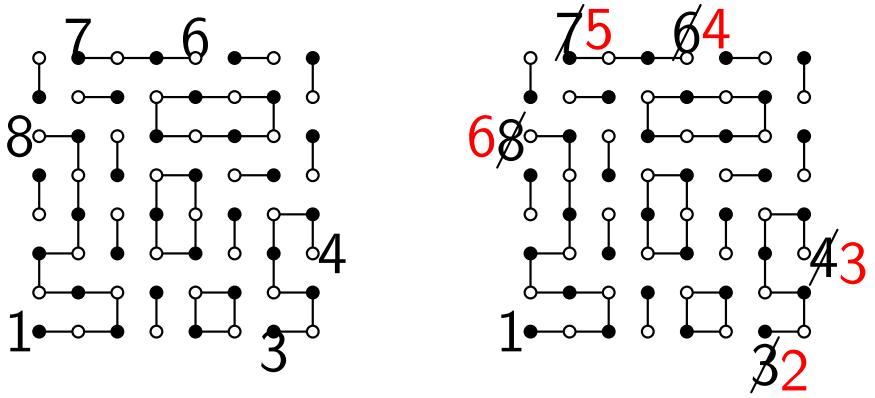
$$\det(M) = \frac{Z_{\sigma}^{DD}(\mathbf{N})}{(Z^D(G^{BW}))^2} \qquad \checkmark$$

$$\det(M_3^3) \stackrel{?}{=} \frac{Z^{DD}_{\sigma_2}(G, \mathbf{N} - \{2,5\})}{(Z^D(G^{BW}))^2}, \text{ where } M_3^3 = \begin{pmatrix} X_{1,8} & X_{1,4} & X_{1,6} \\ X_{3,8} & X_{3,4} & X_{3,6} \\ 0 & X_{7,4} & X_{7,6} \end{pmatrix}$$



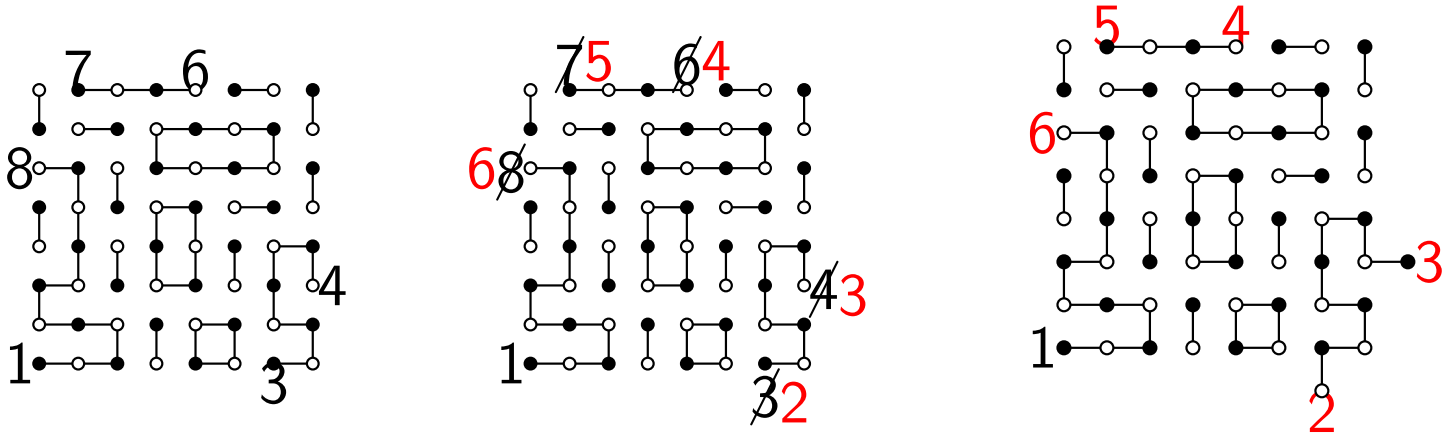
- The nodes are not numbered consecutively.

$$\det(M_3^3) \stackrel{?}{=} \frac{Z_{\sigma_2}^{DD}(G, \mathbf{N} - \{2,5\})}{(Z^D(G^{BW}))^2}, \text{ where } M_3^3 = \begin{pmatrix} X_{1,8} & X_{1,4} & X_{1,6} \\ X_{3,8} & X_{3,4} & X_{3,6} \\ 0 & X_{7,4} & X_{7,6} \end{pmatrix}$$



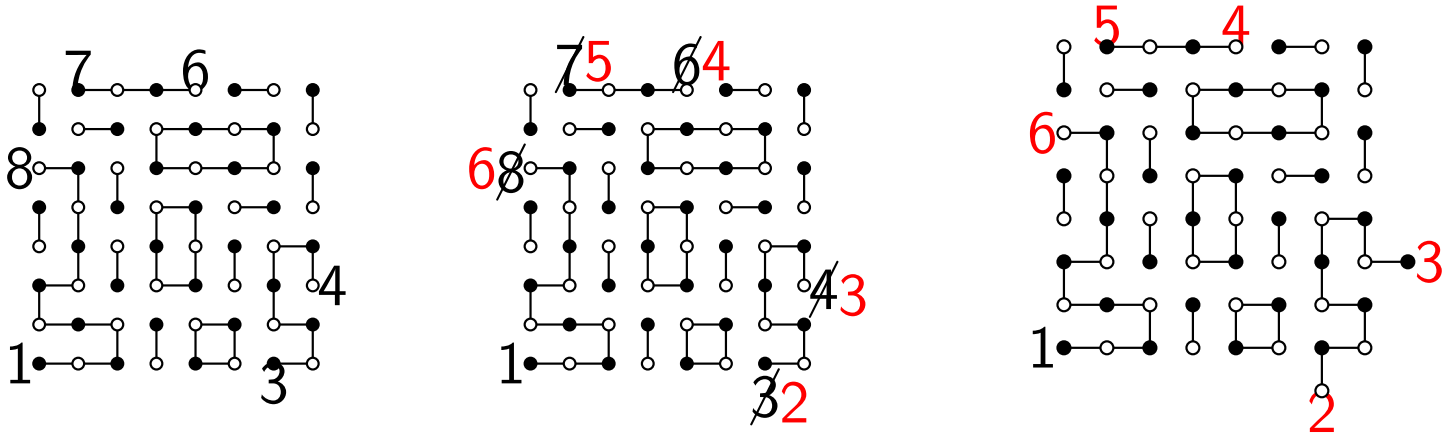
- Relabel the nodes.
- Node 2 is black and node 3 is white.

$$\det(M_3^3) \stackrel{?}{=} \frac{Z^{DD}_{\sigma_2}(G, \mathbf{N} - \{2,5\})}{(Z^D(G^{BW}))^2}, \text{ where } M_3^3 = \begin{pmatrix} X_{1,8} & X_{1,4} & X_{1,6} \\ X_{3,8} & X_{3,4} & X_{3,6} \\ 0 & X_{7,4} & X_{7,6} \end{pmatrix}$$



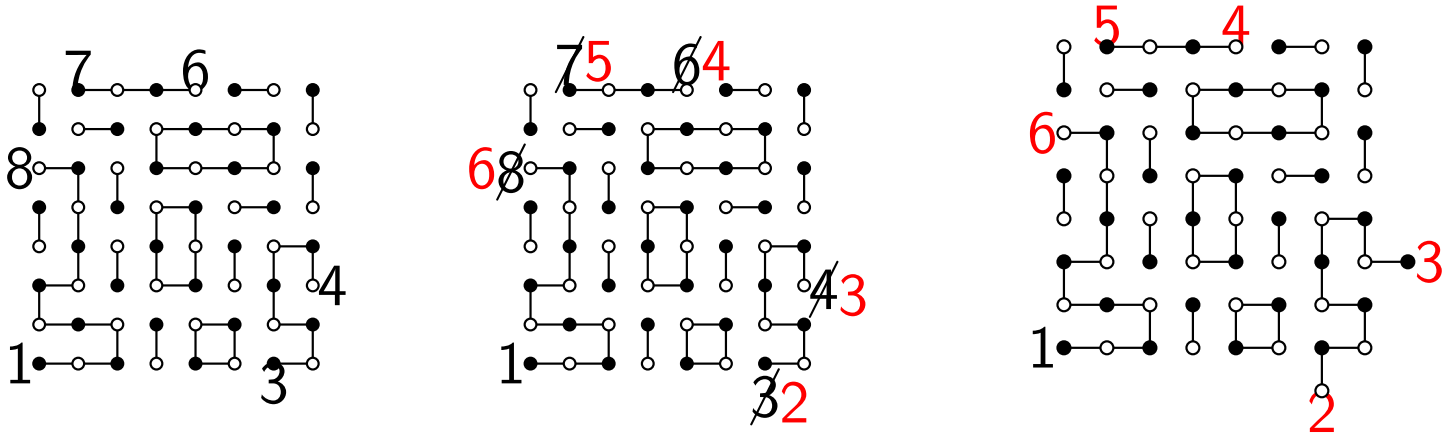
- Add edges of weight 1 to nodes 2 and 3.

$$\det(M_3^3) \stackrel{?}{=} \frac{Z^{DD}_{\sigma_2}(G, \mathbf{N} - \{2,5\})}{(Z^D(G^{BW}))^2}, \text{ where } M_3^3 = \begin{pmatrix} X_{1,8} & X_{1,4} & X_{1,6} \\ X_{3,8} & X_{3,4} & X_{3,6} \\ 0 & X_{7,4} & X_{7,6} \end{pmatrix}$$



- Add edges of weight 1 to nodes 2 and 3.
- The K-W matrix for this new graph will have different entries!

$$\det(M_3^3) \stackrel{?}{=} \frac{Z^{DD}_{\sigma_2}(G, \mathbf{N} - \{2,5\})}{(Z^D(G^{BW}))^2}, \text{ where } M_3^3 = \begin{pmatrix} X_{1,8} & X_{1,4} & X_{1,6} \\ X_{3,8} & X_{3,4} & X_{3,6} \\ 0 & X_{7,4} & X_{7,6} \end{pmatrix}$$

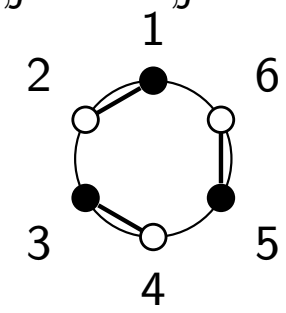


- Add edges of weight 1 to nodes 2 and 3.
- The K-W matrix for this new graph will have different entries!

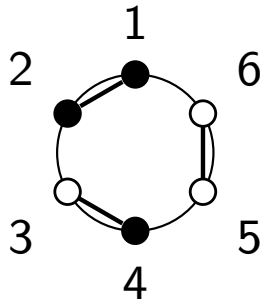
Observation. We need to lift the assumption that the nodes of the graph are black and odd or white and even.

Our Approach

- We establish analogues of K-W without their node coloring constraint.
- Let $Y_{i,j} = \dfrac{Z^D(G_{i,j})}{Z^D(G)}$ and let $\widetilde{\mathrm{Pr}}(\sigma) = \dfrac{Z^{DD}_{\sigma}(G)}{(Z^D(G))^2}$.
- When the nodes are black and odd or white and even, $G = G^{BW}$, so $Y_{i,j} = X_{i,j}$ and $\widetilde{\mathrm{Pr}}(\sigma) = \widehat{\mathrm{Pr}}(\sigma)$.



$$\widehat{\mathrm{Pr}}\left(\begin{array}{c|c|c}1 & 3 & 5\\ 2 & 4 & 6\end{array}\right) = X_{1,4}X_{2,5}X_{3,6} + X_{1,2}X_{3,4}X_{5,6}$$



$$\widetilde{\mathrm{Pr}}\left(\begin{array}{c|c|c}1 & 3 & 5\\ 2 & 4 & 6\end{array}\right) = Y_{1,3}Y_{2,5}Y_{4,6} + Y_{1,5}Y_{2,6}Y_{4,3}$$

Theorem (J.)

$\widetilde{\mathrm{Pr}}(\sigma)$ *is an integer-coefficient homogenous polynomial in* $Y_{i,j}$.

Sign Lemma

Lemma (KW11a, Lemma 3.4)

For odd-even pairings ρ,

$$sign_{OE}(\rho) \prod_{(i,j)\in\rho} (-1)^{(|i-j|-1)/2} = (-1)^{\#\text{ crosses of }\rho}.$$

Lemma (J.)

If ρ is a black-white pairing,

$$sign_c(\mathbf{N}) sign_{BW}(\rho) \prod_{(i,j)\in\rho} sign(i,j) = (-1)^{\#\text{ crosses of }\rho}.$$

Determinant Formula

Theorem (KW09, Theorem 6.1)

When σ is a tripartite pairing,

$$\begin{aligned}\widehat{Pr}(\sigma) &= \det[1_{i,j \text{ RGB-colored differently}} X_{i,j}]_{j=\sigma(1),\sigma(3),\ldots,\sigma(2n-1)}^{i=1,3,\ldots,2n-1} \\ &= sign_{OE}(\sigma) \det[1_{i,j \text{ RGB-colored diff}} X_{i,j}]_{j=2,4,\ldots,2n}^{i=1,3,\ldots,2n-1}\end{aligned}$$

Example ($\text{sign}_{OE}(\sigma)$)

If $\sigma = \left(\begin{array}{c|c|c} 1 & 3 & 5 \\ 6 & 2 & 4 \end{array}\right)$, then $\text{sign}_{OE}(\sigma)$ is the parity of $\begin{pmatrix} \frac{6}{2} & \frac{2}{2} & \frac{4}{2} \end{pmatrix} = \begin{pmatrix} 3 & 1 & 2 \end{pmatrix}$

Determinant Formula

Theorem (KW09, Theorem 6.1)

When σ is a tripartite pairing,

$$\begin{aligned}\widehat{Pr}(\sigma) &= \det[1_{i,j \text{ RGB-colored differently}} X_{i,j}]_{j=\sigma(1),\sigma(3),\ldots,\sigma(2n-1)}^{i=1,3,\ldots,2n-1} \\ &= sign_{OE}(\sigma) \det[1_{i,j \text{ RGB-colored diff}} X_{i,j}]_{j=2,4,\ldots,2n}^{i=1,3,\ldots,2n-1}\end{aligned}$$

Example ($\mathrm{sign}_{OE}(\sigma)$)

If $\sigma = \left(\begin{array}{c|c|c} 1 & 3 & 5 \\ 6 & 2 & 4 \end{array}\right)$, then $\mathrm{sign}_{OE}(\sigma)$ is the parity of $\begin{pmatrix} \frac{6}{2} & \frac{2}{2} & \frac{4}{2} \end{pmatrix} = \begin{pmatrix} 3 & 1 & 2 \end{pmatrix}$

Theorem (J.)

When σ is a tripartite pairing,

$$\widetilde{Pr}(\sigma) = sign_{OE}(\sigma) \det[1_{i,j \text{ RGB-colored differently}} Y_{i,j}]_{j=w_1,w_2,\ldots,w_n}^{i=b_1,b_2,\ldots,b_n}.$$

More general result

Theorem (J.)

Divide $\mathbf{N}$ *into sets* R, G, *and* B *and let* σ *be the corresponding tripartite pairing. If* $x, w \in V_1$ *and* $y, v \in V_2$ *then*

$$\begin{aligned} & sign_{OE}(\sigma) sign_{OE}(\sigma_5) Z^{DD}_{\sigma}(\mathbf{N}) Z^{DD}_{\sigma_5}(\mathbf{N} - \{x, y, w, v\}) \\ = \; & sign_{OE}(\sigma_1) sign_{OE}(\sigma_2) Z^{DD}_{\sigma_1}(\mathbf{N} - \{x, y\}) Z^{DD}_{\sigma_2}(\mathbf{N} - \{w, v\}) \\ & - sign_{OE}(\sigma_3) sign_{OE}(\sigma_4) Z^{DD}_{\sigma_3}(\mathbf{N} - \{x, v\}) Z^{DD}_{\sigma_4}(\mathbf{N} - \{w, y\}) \end{aligned}$$

More general result

Theorem (J.)

Divide $\mathbf{N}$ *into sets* R, G, *and* B *and let* σ *be the corresponding tripartite pairing. If* $x, w \in V_1$ *and* $y, v \in V_2$ *then*

$$\begin{aligned} & sign_{OE}(\sigma) sign_{OE}(\sigma_5) Z_{\sigma}^{DD}(\mathbf{N}) Z_{\sigma_5}^{DD}(\mathbf{N} - \{x, y, w, v\}) \\ = \; & sign_{OE}(\sigma_1) sign_{OE}(\sigma_2) Z_{\sigma_1}^{DD}(\mathbf{N} - \{x, y\}) Z_{\sigma_2}^{DD}(\mathbf{N} - \{w, v\}) \\ & - sign_{OE}(\sigma_3) sign_{OE}(\sigma_4) Z_{\sigma_3}^{DD}(\mathbf{N} - \{x, v\}) Z_{\sigma_4}^{DD}(\mathbf{N} - \{w, y\}) \end{aligned}$$

Corollary

Divide $\mathbf{N}$ *into sets* R, G, *and* B *and let* σ *be the corresponding tripartite pairing. Let* $x, y, w, v \in \mathbf{N}$ *appear in a cyclic order such that* $\{x, y, w, v\}$ *contains at least one node of each RGB color. If* $x, w \in V_1$ *and* $y, v \in V_2$,

$$\begin{aligned} & Z_{\sigma}^{DD}(\mathbf{N}) Z_{\sigma_5}^{DD}(\mathbf{N} - \{x, y, w, v\}) \quad = \\ & Z_{\sigma_1}^{DD}(\mathbf{N} - \{x, y\}) Z_{\sigma_2}^{DD}(\mathbf{N} - \{w, v\}) + Z_{\sigma_3}^{DD}(\mathbf{N} - \{x, v\}) Z_{\sigma_4}^{DD}(\mathbf{N} - \{w, y\}) \end{aligned}$$

Thank you for listening!

References

- Noam Elkies, Greg Kuperberg, Michael Larsen, and James Propp. Alternating-Sign matrices and Domino Tilings (Part I). *J. Algebraic Combin.* 1(2):111-132, 1992.
- Markus Fulmek, Graphical condensation, overlapping Pfaffians and superpositions of matchings. *Electron. J. Comb.*, 17, 2010.
- Helen Jenne. Combinatorics of the double-dimer model. *arXiv preprint arXiv:1911.04079*, 2019.
- Richard W. Kenyon and David B. Wilson. Combinatorics of tripartite boundary connections for trees and dimers. *Electron. J Comb.*, 16(1), 2009.
- Richard W. Kenyon and David B. Wilson. Boundary partitions in trees and dimers. *Trans. Amer. Math. Soc.*, 363(3):1325-1364, 2011.
- Eric H Kuo. Applications of graphical condensation for enumerating matchings and tilings. *Theoret. Comput. Sci.*, 319(1-3):29-57, 2004.
- Tri Lai and Gregg Musiker. Beyond Aztec castles: toric cascades in the dP_3 quiver. *Comm. Math. Phys.*, 356(3):823-881, 2017.
- David E Speyer. Variations on a theme of Kasteleyn, with Application to the Totally Nonnegative Grassmannian. *Electron. J. Comb.*, 23(2), 2016.