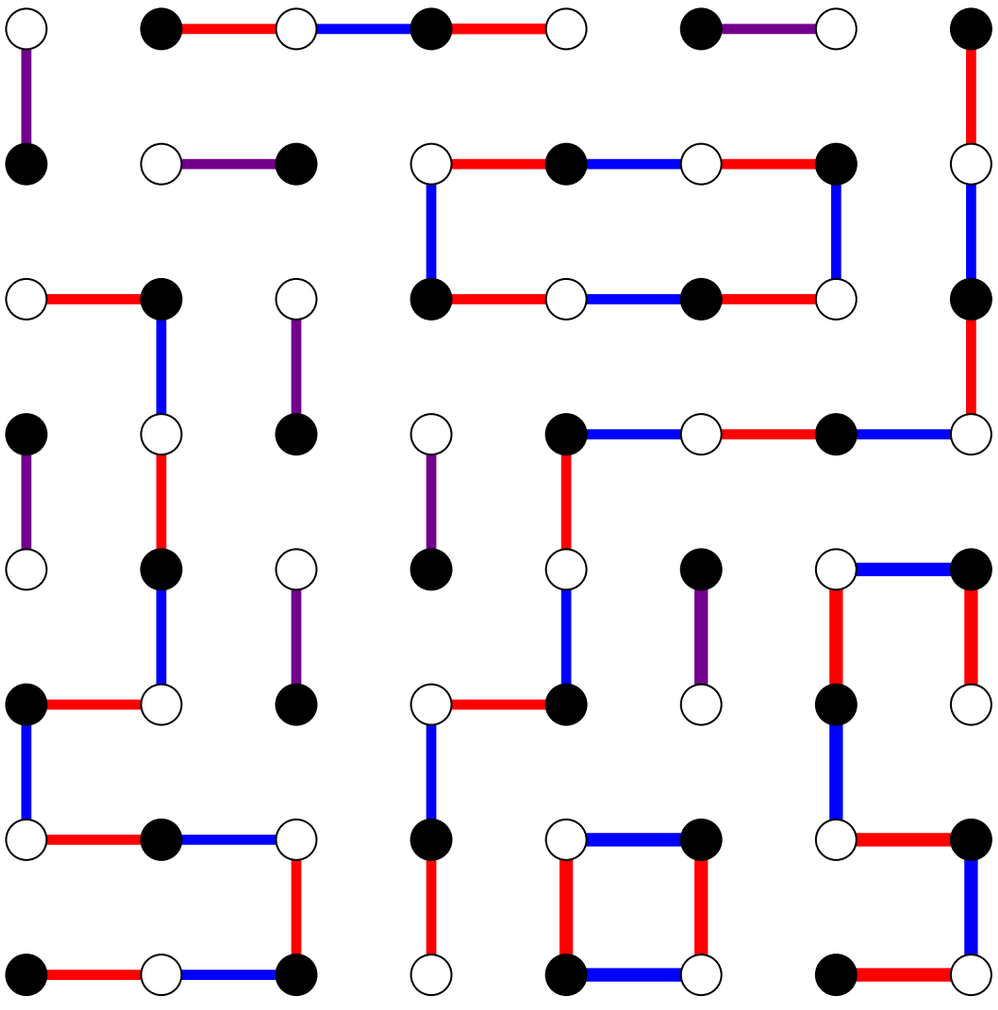
Combinatorics of the Double-Dimer Model



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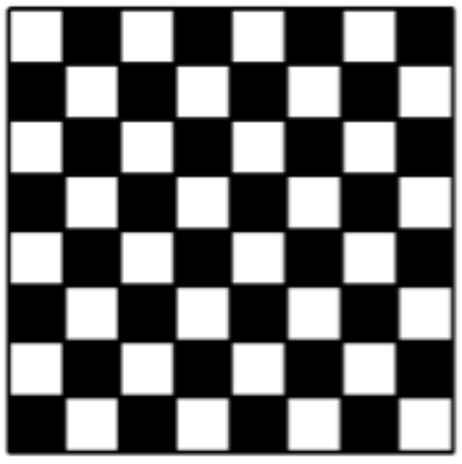
What is Combinatorics?

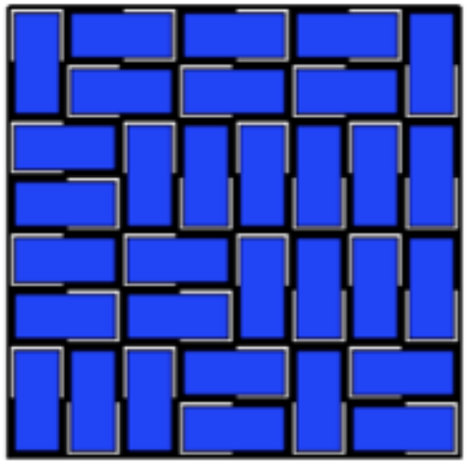
Combinatorialists like to ask:

- Does such-and-such exist?
- If it does, how many are there?

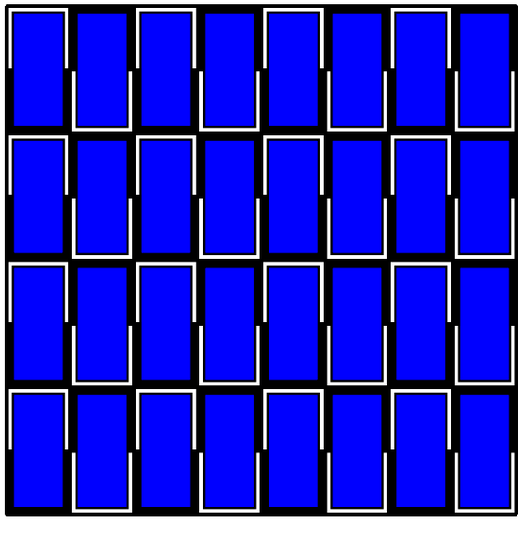
Example question: Can we arrange dominoes on a standard chessboard so that

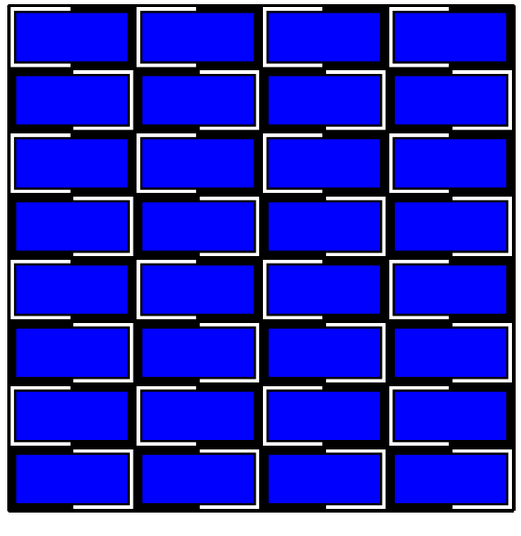
- no two dominoes overlap,
- every domino covers two squares, and
- all of the squares are covered?





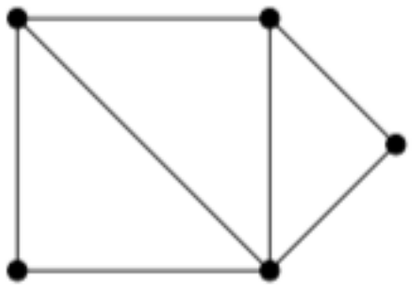
How many arrangements are there? 12,988,816



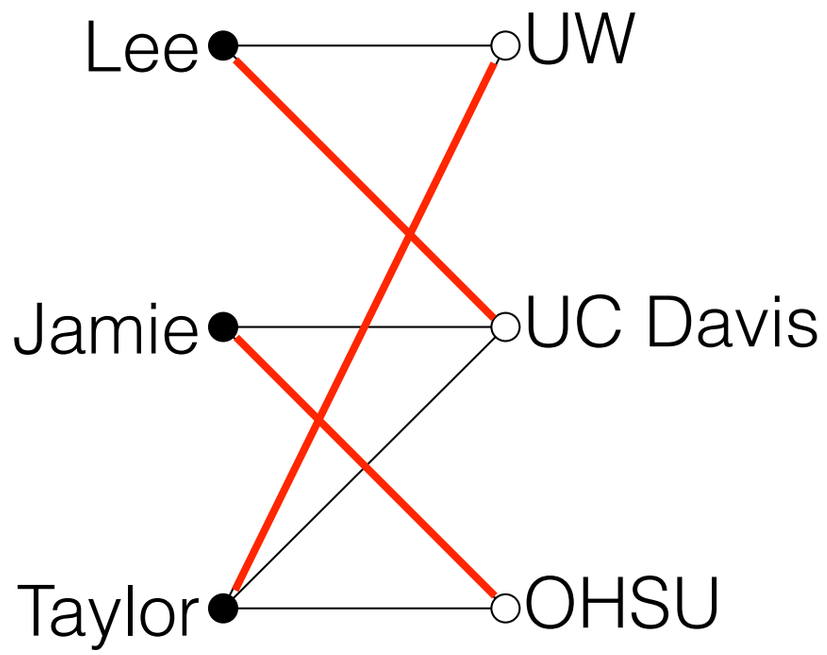


By answering a similar question, in the 1960's Kasteleyn gave an elegant solution for the *honeycomb lattice Ising model*.

Rephrasing in the language of graphs



A *graph* is made up of vertices (points) which are connected by edges (links)



Kasteleyn studied *dimer configurations* of graphs.

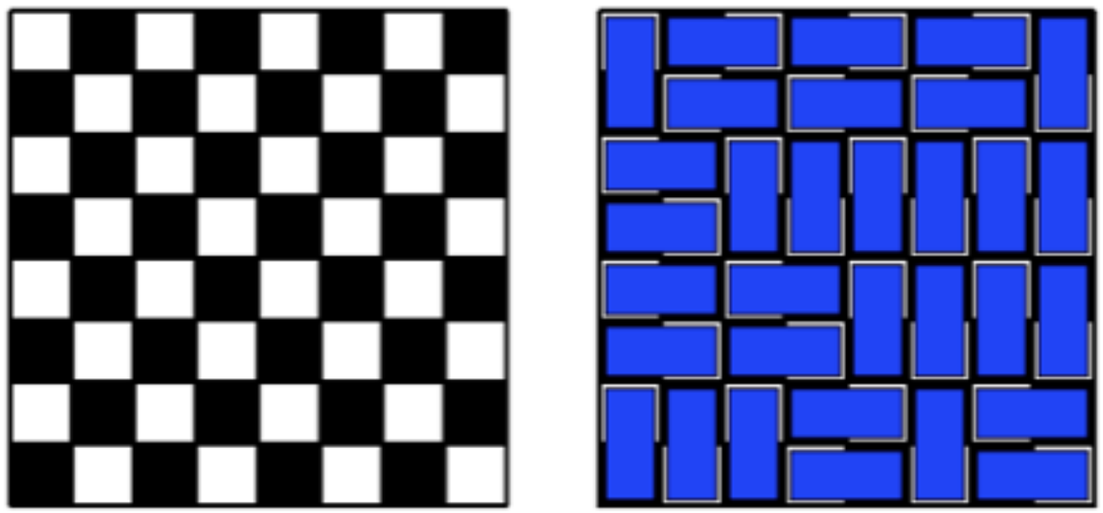
Example: The *vertices* consist of medical students and residency programs. There is an *edge* between a medical student vertex and a program vertex if they are interested in each other.

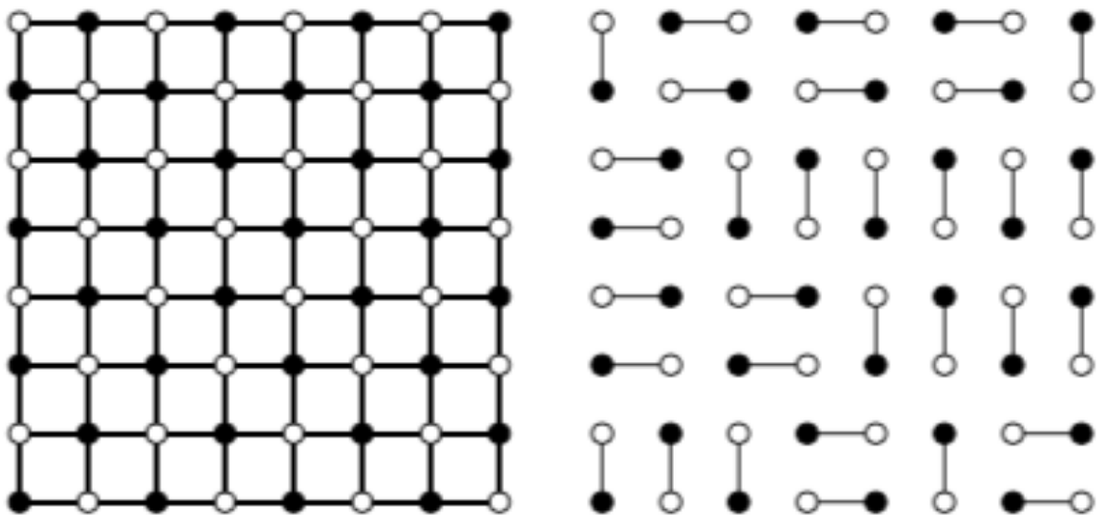
A *dimer configuration* is an assignment of each student to exactly one residency program.

More generally, a dimer configuration of a graph is a selection of edges that connects each vertex to exactly one other vertex.

Rephrasing in the language of graphs

Counting domino tilings of a chessboard is equivalent to counting *dimer configurations* of a grid graph.





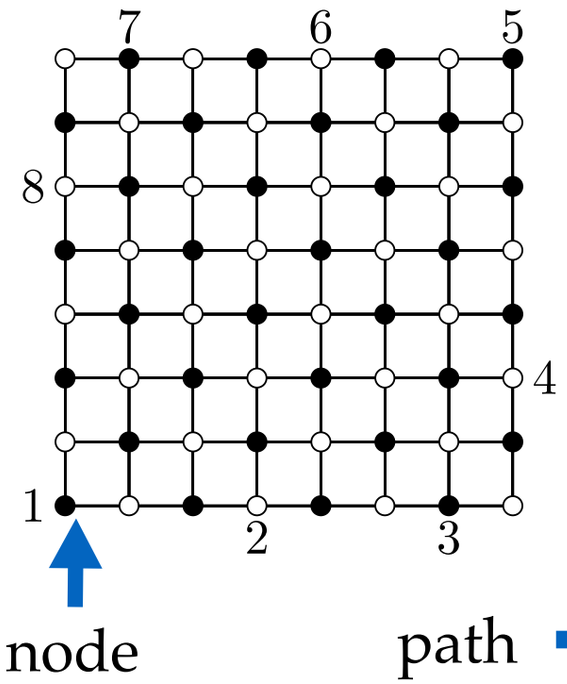
Original question:

How many domino coverings of a chessboard are there?

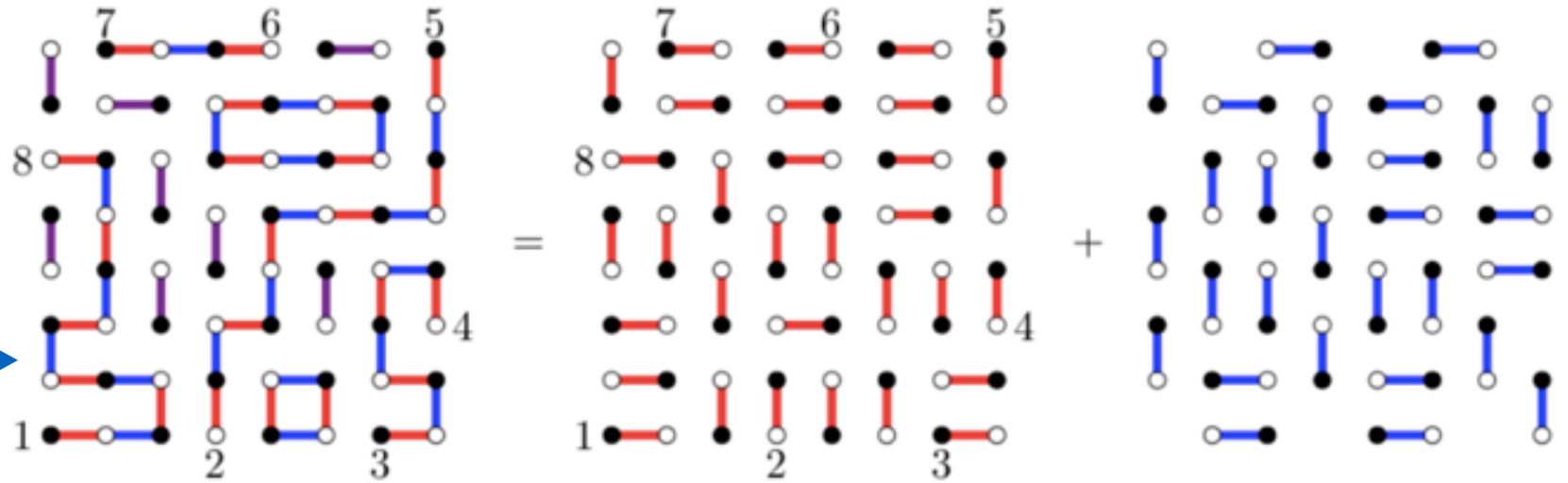
Equivalent question:

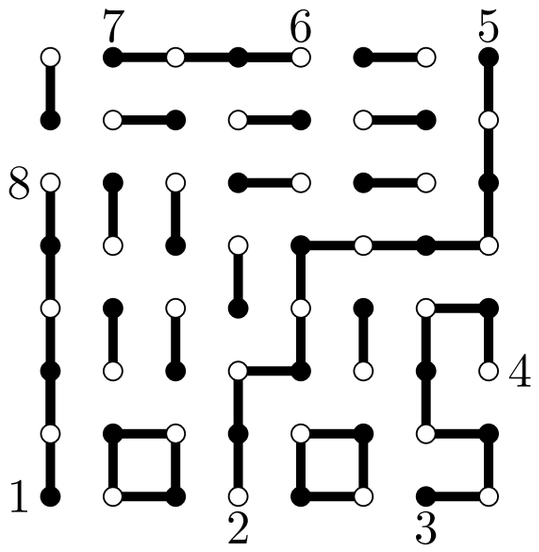
How many dimer configurations of an 8 x 8 grid graph are there?

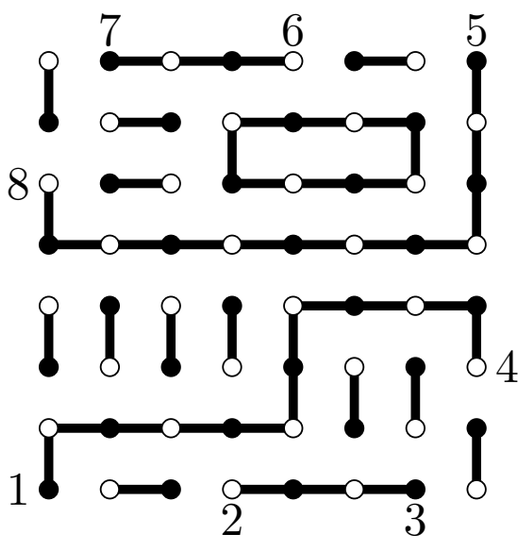
What I study: Double-dimer configurations



A *double-dimer configuration* of a graph is two dimer configurations.







How can we count the number of double-dimer configurations that have particular characteristics?

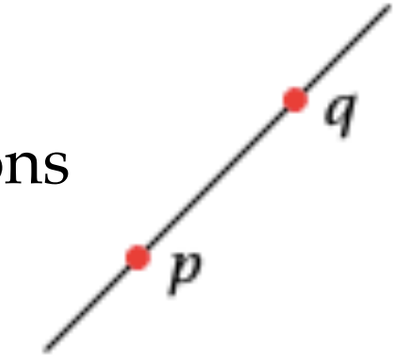
Applications to other areas of math

Building on the work of Kenyon and Wilson, I proved that under certain conditions the number of double-dimer configurations satisfies a *recurrence*.

My result helps count double-dimer configurations, and has applications to problems in other areas of math!

Enumerative geometry

Counts geometric objects that satisfy certain geometric conditions



- How many lines pass through 2 points in the plane?
- How many lines pass through 4 lines in three dimensional space?

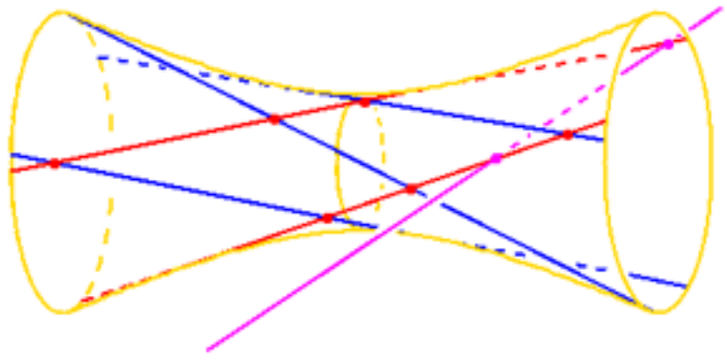


Image credits: Ricolfi, A. "Introduction to Enumerative Geometry."; Sottile, F. "The two lines meeting 4 lines in 3-space."

Thank you!

References

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- R. W. Kenyon and D. B. Wilson. Boundary partitions in trees and dimers. *Trans. Amer. Math. Soc.*, 363(3):1325-1364, 2011.
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