The Dimer Model and Kuo Condensation

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UW Combinatorics Pre-Seminar

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This talk is being recorded

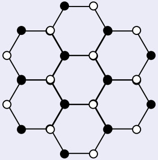
Outline

1. Introduction
2. Kuo condensation
3. Double-dimer configurations

Dimer configurations

Definition

A graph is *bipartite* if its vertices can be colored black and white so that every edge connects a black vertex to a white vertex.



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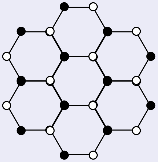

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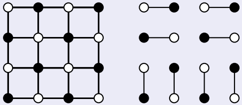
Definition (Dimer configuration/Perfect matching)



A collection of edges that covers each vertex exactly once

Throughout this talk, we'll assume $|V_1| = |V_2|$.

Dimer configurations

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Theorem (Kas67)

If G is a bipartite planar graph, there is a matrix K with the property that $\det(K)$ is the number of dimer configurations of G.

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The number of dimer configurations of an $m \times n$ grid graph is

$$2^{mn/2} \prod_{k=1}^{m/2} \prod_{\ell=1}^{n} \left(\cos^2 \frac{\pi k}{m+1} + \cos^2 \frac{\pi \ell}{n+1} \right)^{1/2}$$

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Excellent survey of tools for counting dimer configurations: Enumeration of Tilings by Jim Propp.

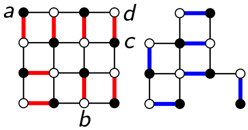
Kuo condensation

Let $M(G)$ denote the number of perfect matchings of G.

Theorem (Kuo04, Theorem 2.1)

Let vertices a, b, c, and d appear in a cyclic order on a face of G. If $a, c \in V_1$ and $b, d \in V_2$, then

$$M(G)M(G-\{a,b,c,d\})=M(G-\{a,b\})M(G-\{c,d\})+M(G-\{a,d\})M(G-\{b,c\})$$



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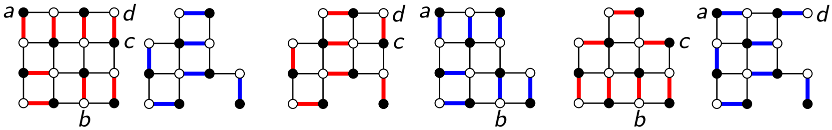

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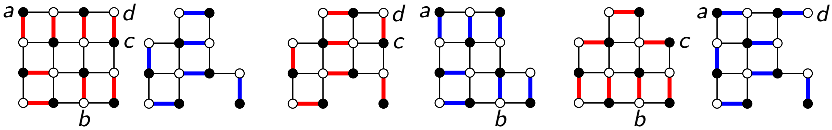

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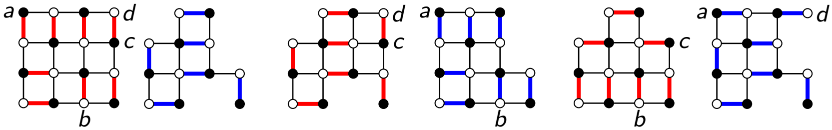

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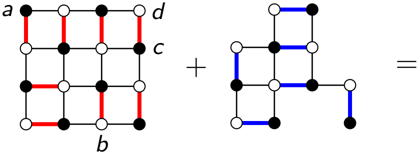
Goal. Show $|\mathcal{M}(G) \times \mathcal{M}(G-\{a,b,c,d\})| = |\mathcal{M}(G-\{a,b\}) \times \mathcal{M}(G-\{c,d\}) \cup \mathcal{M}(G-\{a,d\}) \times \mathcal{M}(G-\{b,c\})|$

Proof of Kuo's Theorem

- Superimpose matchings of G and $G - \{a, b, c, d\}$.



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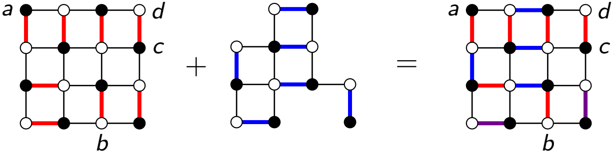

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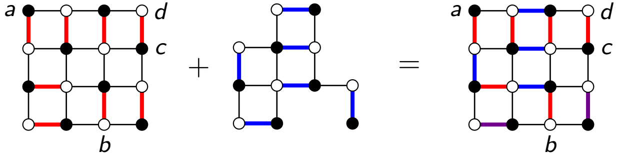

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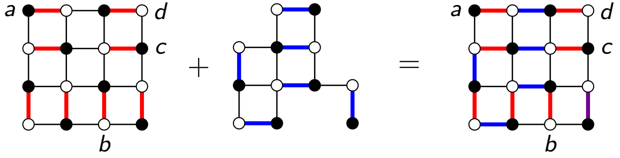

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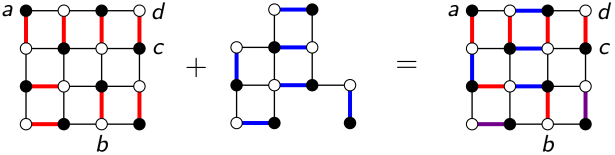

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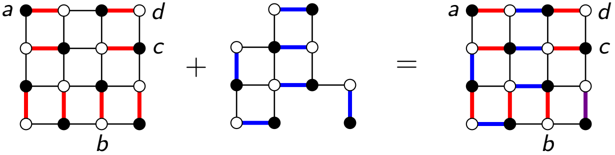

- The multigraph will contain paths with endpoints a, b, c, and d.

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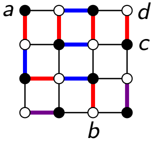

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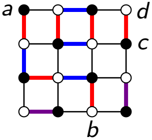

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We will show:

Claim 1. H can be partitioned into a matching of G and a matching of $G - \{a, b, c, d\}$ in 2^c ways, where c is the number of cycles.

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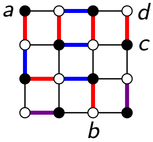

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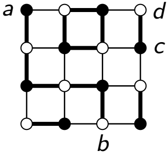

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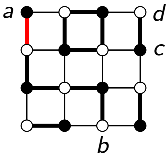

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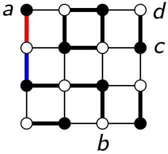

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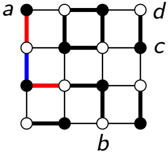

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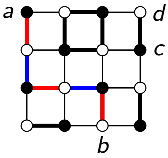

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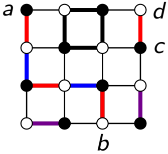

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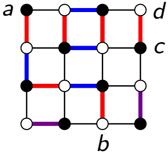

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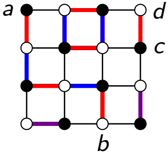

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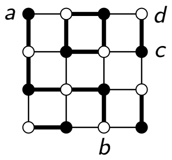

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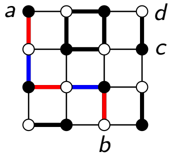

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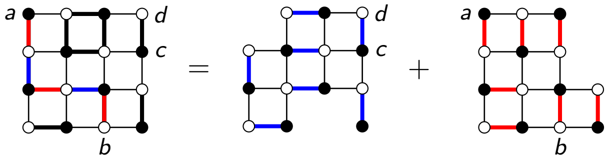

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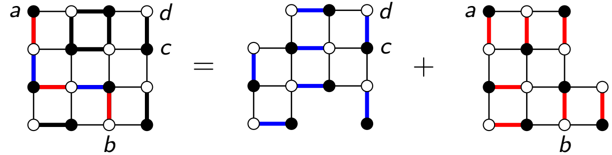
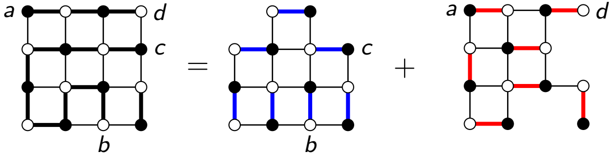
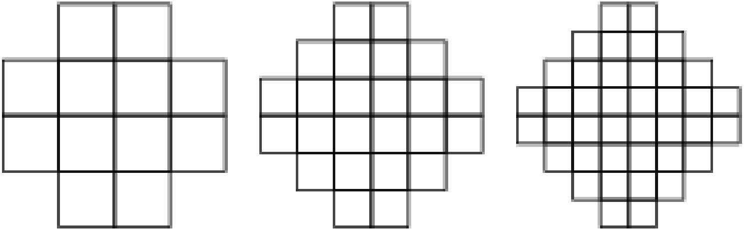




Definition

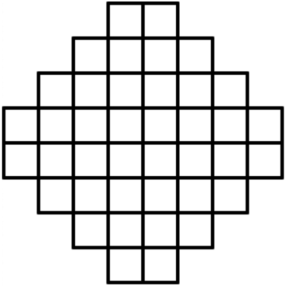
An Aztec diamond of order n consists of $2n$ centered rows of unit squares of lengths $2, 4, \ldots, 2n-2, 2n, 2n, 2n-2, \ldots, 4, 2$.

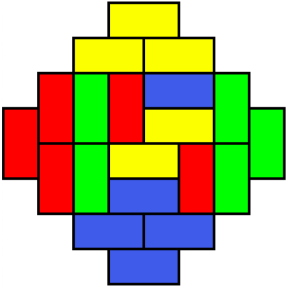


order 2 order 3 order 4

Image credit: WolframMathWorld

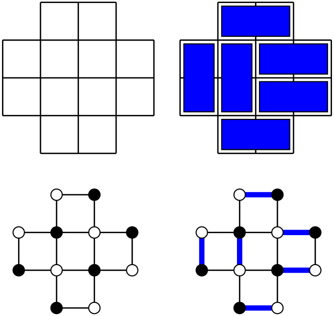
Application: The Aztec Diamond Theorem





Theorem (EKLP92)

The number of tilings of the order-n Aztec diamond is $2^{n(n+1)/2}$.



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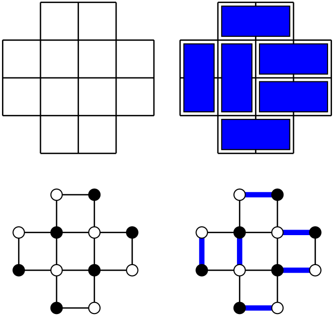

Theorem (Kuo04, Prop 3.1)

Let $T(n)$ denote the number of matchings of an Aztec diamond graph of order-n. Then

$$T(n)T(n-2) = 2(T(n-1))^2.$$

Edge-weighted graphs

- Given a graph $G = (V, E)$, we can assign a weight $w(e)$ to each edge.
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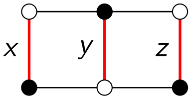

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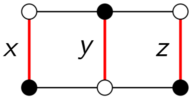

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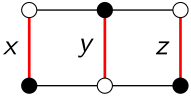

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$$Z^D(G) = xyz + x + z$$

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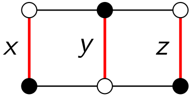

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Theorem (Kuo04, Theorem 5.1)

Let vertices a, b, c, and d appear in a cyclic order on a face of G. If $a, c \in V_1$ and $b, d \in V_2$, then

$$Z^D(G)Z^D(G-\{a,b,c,d\})=Z^D(G-\{a,b\})Z^D(G-\{c,d\})+Z^D(G-\{a,d\})Z^D(G-\{b,c\})$$

Double-dimer configurations

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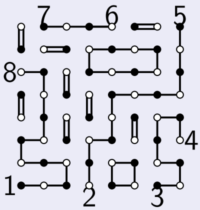

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- Doubled edges
- Paths connecting nodes in pairs

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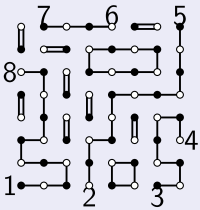

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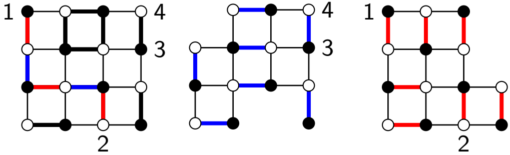
Its weight is the product of its edge weights $\times 2^{\ell}$

Double-dimer configurations

Let $Z_\sigma^{DD}(G, \mathbf{N})$ denote the weighted sum of all DD configs with pairing σ.

By Kuo's proof,

$$Z^{DD}_{((1,2),(3,4))}(G, \mathbf{N}) = Z^D(G - \{1,2\})Z^D(G - \{3,4\})$$

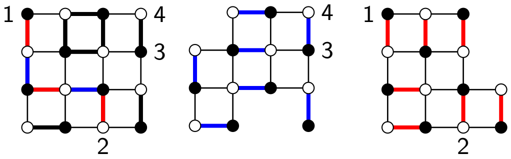


Double-dimer configurations

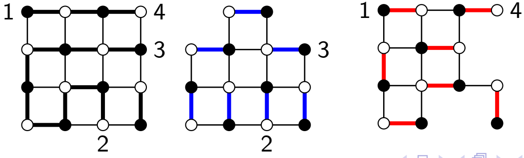
Let $Z^{DD}_{\sigma}(G, \mathbf{N})$ denote the weighted sum of all DD configs with pairing σ.

By Kuo's proof,

$$Z^{DD}_{((1,2),(3,4))}(G, \mathbf{N}) = Z^{D}(G - \{1,2\})Z^{D}(G - \{3,4\})$$



$$Z^{DD}_{((1,4),(3,2))}(G, \mathbf{N}) = Z^{D}(G - \{1,4\})Z^{D}(G - \{2,3\})$$



Double-dimer configurations

$$Z^{DD}_{((1,2),(3,4))}(G, \mathbf{N}) = Z^D(G - \{1,2\})Z^D(G - \{3,4\})$$

$$Z^{DD}_{((1,4),(3,2))}(G, \mathbf{N}) = Z^D(G - \{1,4\})Z^D(G - \{2,3\})$$

Let $X_{i,j} = \dfrac{Z^D(G - \{i,j\})}{Z^D(G)}$. Then the above two equations become...

$$\frac{Z^{DD}_{((1,2),(3,4))}(G, \mathbf{N})}{(Z^D(G))^2} = X_{1,2}X_{3,4}$$

$$\frac{Z^{DD}_{((1,4),(3,2))}(G, \mathbf{N})}{(Z^D(G))^2} = X_{1,4}X_{2,3}$$

Thank you for listening!

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